STABILITY OF JACKSON-TYPE QUEUEING NETWORKS, I

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Abstract

This paper gives a pathwise construction of Jackson-type queueing networks allowing the derivation of stability and convergence theorems under general probabilistic assumptions on the driving sequences; namely, it is only assumed that the input process, the service sequences and the routing mechanism are jointly stationary and ergodic in a sense that is made precise in the paper. The main tools for these results are the subadditive ergodic theorem, which is used to derive a strong law of large numbers, and basic theorems on monotone stochastic recursive sequences. The techniques which are proposed here apply to other and more general classes of discrete event systems, like Petri nets or GSMP's. The paper also provides new results on the Jackson-type networks with i.i.d. driving sequences which were studied in the past.

Keywords: Ordered directed graph, Euler graphs, Euler ordered directed graph, switching sequence, open Jackson-type queueing network, point processes, Euler network, composition, decomposition, conservation rule, departure and throughput processes, first and second-order ergodic properties, subadditive ergodic theorem, solidarity property, stochastic recursive sequences, stationary solution, coupling-convergence, uniqueness of the stationary regime.

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1 Introduction

A class of queueing systems is often considered as well understood if its state can be constructed and its properties analyzed under general statistical assumptions, namely stationarity and ergodicity assumptions on the data of the system under consideration (see Loynes [29], Borovkov [8], Franken, König, Arndt and Schmidt [21], Baccelli and Brémaud [2], Brandt, Franken and Lisek [15], Borovkov and Foss [12]). Such constructions and analysis have been available for quite general classes of acyclic queueing networks (see Kalashnikov and Rachev [24], Konstantopoulos and Walrand [27] for instance), but only for specific classes of cyclic networks (see Baccelli and Liu [6], Baccelli, Cohen, Olsder and Quadrat [5], Afanas'eva [1]). Most of these contributions are based on pathwise recursions which can be traced back to the pioneering work of Loynes [29].

To the best of our knowledge, for Jackson-type networks, the stability problem was only approached either under specific statistical assumptions (this is the case for their definition using product form theory by Jackson [22]), or under certain modifications of the service mechanism (see for instance Baccelli and Liu [6] and Afanas'eva [1], who introduce either synchronization constraints or priorities in order to analyze the network). Although some of the models focusing on the actual Jackson-type problem are rather general (see Borovkov [10], Foss [18]-[19], Kumar and Meyn [28] and Konstantopoulos and Walrand [27] for instance), all of them require some sort of independence property or some distributional restrictions (see Foss [19] for a partial bibliography on the matter). More generally, for the type of general assumptions alluded to above (stationary-ergodic), no construction of the state of the network providing ergodic theorems seems to be currently available. The object of the present paper is to make such a construction.

A first difficulty arises with the pathwise definition of such a generalized non-Markovian queueing network. The networks we consider in this paper are characterized by the fact that service times and switching decisions are associated with *stations*, and not with *customers*. This means that the j-th service on station k takes σ_j^k units of time, where $\{\sigma_j^k\}_{j\geq 0}$ is a predefined sequence. In the same way, when this service is completed, the leaving customer is sent to station ν_j^k (or leaves the network) and it is put at the end of the queue on this station, where $\{\nu_j^k\}_{j\geq 0}$ is also a predefined sequence, called the switching sequence. The sequences $\{\sigma_j^k\}_{j\geq 0}$ and $\{\nu_j^k\}_{j\geq 0}$, where k ranges over the set of stations are called the *driving sequences* of the net, and it is on these driving sequences that the statistical assumptions are actually made. This is in a sense the proper generalization of what happens in Markovian Jackson queueing networks where customers sample there service times upon their arrival in a queue and flip a coin locally to determine where to go next. This explains why we propose to call Jackson-type networks those networks with a pathwise definition based on such a *station-centered* numbering scheme.

This pathwise definition has to be opposed to what happens in Kelly-type networks where routing (and service times) are associated with input customers (e.g. an arriving customer has a predefined route and predefined service requirements at each of the stations of its route, all of which are known upon its arrival). This second scheme will be referred to as *customer-centered*.

The distinction between Jackson-type nets, with station-centered numbering and Kelly-type customer-centered nets is quite essential for the purposes of the present paper. The station-centered definition preserves various basic monotonicity properties as already shown in Foss [18]- [19] and Shanthikumar and Yao [33], whereas the second one does not. More importantly, the natural stability condition which we prove to hold in the present paper for Jackson-type station-centered networks was recently shown not to be sufficient for the case of Kelly-type networks based on the customer-centered scheme (see Bramson [14]).

The second difficulty lies in the construction of state variables amenable to some sort of stochastic recurrence equation which satisfies a first-order ergodic theorem (i.e. a SLLN). These variables will be referred to as first-order state variables. The possibility of defining such state variables is obtained from recursive equations which were derived for a class of stochastic Petri net which contains Jackson networks (see Baccelli, Cohen and Gaujal [4]). The understanding of the appropriate stationarity and ergodicity assumptions to be made on service and routing sequences comes from graph theoretic considerations, and in particular from the notion of Euler switching and Euler network. These graph theoretic considerations provide an in-depth understanding on the pathwise dynamics of such networks (see the appendix on the geometry of routes). They also reveal the right ergodicity assumptions to be made on the driving sequences. The first-order ergodic theorem follows from the subadditive property satisfied by the time to clear the system of its workload after the last epoch of an interrupted arrival point process (§4). This technique generalizes that of Baccelli, Cohen, Olsder and Quadrat [5] for the stability of event graphs. The time to clear the workload is an adequate variable for getting a SLLN under rather general assumptions since this is true for event graphs, for Jackson networks, and for certain classes of stochastic Petri nets with general topology (Baccelli and Foss [7]).

The third difficulty comes from the search for increments of the first-order state variables which satisfy some stochastic recurrence equation with appropriate monotonicity properties and for which could be proved a second-order ergodic result of the Loynes-type (e.g. coupling with a stationary ergodic regime or simply weak convergence to such a regime). These second-order state variables are introduced in §3. following ideas developed in Foss [19]. The relation between the finiteness of the second-order state variables and the constants that show up in the SLLN is investigated in § 5. This gives the stability threshold ensuring the finiteness of queue length and the like. The stochastic recursion that these second-order variables satisfy is investigated in §6 and used in §7 for proving certain coupling convergence results and uniqueness results.

Besides the theoretical interest of this construction, several new results or new proofs of known results can be obtained for various models. For instance, we can always construct a minimal stationary regime (see §6). In the particular case when routing and services are i.i.d., the distinction between service associated with stations and service associated with customers vanishes as both coincide in law. So, when restricted to the i.i.d. case, our results show that the Cramer-type conditions considered in Borovkov [10] on the distribution functions of the service times can be relaxed (see also Foss [19] and Chang [16]). Similarly, whenever the switching decisions are i.i.d, we show the following generalization of results in Foss [19]: there is a unique stationary regime which is reached with coupling, under general assumptions on the arrival and service processes.

2 Ordered Directed Graphs.

2.1 Routes and Switching Sequences.

Let K and φ be two positive integers.

Definition 1 *The finite sequence of integers $r = (r_1, \ldots, r_\varphi)$ is a route with length φ on nodes $\{0, 1, \ldots, K, K+1\}$ if $0 \le r_1 \le K$, $1 \le r_i \le K$ for all $i = 2, \ldots, \varphi - 1$ and $1 \le r_\varphi \le K + 1$.*

The variable r_i gives the identity of the i-th node of the route. Node 0 represents the source and node $K+1$ represents the sink. Thus, a route can start either from the source of from an internal node, and it must stop either in the sink or in some internal node. The variable φ will be referred to as the *length* of the route. For reasons which will become clear later, it is often useful to consider nodes 0 and $K+1$ as a single node, and we will do it without special warning when this is non-ambiguous.

Definition 2 *The route r is* admissible *if $r_\varphi \in \{r_1, K+1\}$ and* successful *if in addition $r_\varphi = K+1$, $r_1 = 0$. If $r_\varphi = r_1 \neq 0$, r will also be called a* circuit. *This circuit will be said to be* simple *if it contains no other smaller circuit.*

Consider an admissible route r with length φ. For each $k, l = 0, 1, \ldots, K, K+1$, let

$$\varphi^{k,l} = \sharp\{i : r_i = k,\ r_{i+1} = l\}, \tag{1}$$

and let

$$\varphi^k = \sum_{l=1}^{K+1} \varphi^{k,l}. \tag{2}$$

The variable φ^k counts the number of departures from node k. If route r is admissible, then for each node $k = 1, \ldots, K$, the number of arrivals to node k should be equal to the number of departures from k and therefore

$$\varphi^k = \sum_{l=0}^{K} \varphi^{l,k}, \tag{3}$$

for all $k = 1, \ldots, K$. Thus, for a successfull route,

$$\sum_{l=1}^{K+1} \varphi^{k,l} = \sum_{l=0}^{K} \varphi^{l,k} \quad \forall k = 1, \ldots, K, \tag{4}$$

and in addition

$$\sum_{l=1}^{K+1} \varphi^{0,l} = \sum_{l=0}^{K} \varphi^{l,K+1} = 1. \tag{5}$$

With a route r and node $k \in \{0, 1, \ldots, K\}$, we associate a $\{1, \ldots, K, K+1\}$-valued sequence ν^k which describes the successive switching decisions from node k and which is defined as follows:

- for $k = 0, 1, \ldots, K+1$, if $\varphi^k = 0$, then $\nu^k = \emptyset$ (i.e. ν^k is the empty sequence);
- for $k = 0, 1, \ldots, K, K+1$, if $\varphi^k > 0$ then
  - Consider the auxiliary sequence $\{q_n^k\}_{n=1}^{\varphi^k}$ giving the successive visit times to node k, defined by $q_1^k = \min\{i \geq 1 : r_i = k\}$ and for $n = 1, \ldots, \varphi^k - 1$ by $q_{n+1}^k = \min\{i \geq q_n^k + 1 : r_i = k\}$;
  - For $n = 1, \ldots, \varphi^k$ let $\nu_n^k = r_{q_n^k+1}$.

Then ν^k is the sequence $\nu^k = \{\nu_1^k, \ldots, \nu_{\varphi^k}^k\}$. We shall say that ν^k is the *switching sequence* of node k *generated* by route r, and that $\nu = \{\nu^k\}_{k=0}^K$ is the *switching sequence generated by route* r. This switching sequence will be said to be simple because it is generated by a single route.

Conversely, define a switching sequence ν on nodes $\{0, 1, \ldots, K, K+1\}$ to be a family of finite sequences $\{\nu_j^k\}_{j=1}^{d^k}$, $k = 0, \ldots, K+1$, where ν_j^k belongs to $\{1, \ldots, K+1\}$. By definition, $d^{K+1} = 0$. The switching sequence is simple if in addition $d^0 = 1$. Consider the following procedure:

Procedure 1 Path (G, k)

$l_1 := k$; $m^p := 0\ \forall p = 0, \ldots, K+1$; $t := 1$;

while $m^{l_t} < d^{l_t}$ **do**

begin

- $m^{l_t} := m^{l_t} + 1$;
- $l_{t+1} := \nu_{m_{l_t}}^{l_t}$;
- $t := t + 1$;

end

By definition, the *path* originating from node k generated by the switching sequence ν is the sequence $l_1, l_2, \ldots$ produced by this procedure. This path is a finite, non-necessarily admissible route. We will say that it is *exhaustive* if the value of the variable m^k when the procedure stops, say Φ^k (not to be confused with φ^k) is equal to d^k for all k. It may happen that the produced path is not exhaustive.

Remark 1 *If the switching sequence ν, with length d^k on node k is that generated by a finite and successful route r with parameter φ^k on node k, then the path generated by ν and originating from* 0 *is also route r; this path is exhaustive, and thus $\Phi^k = d^k = \varphi^k$ for all k.*

However, we can consider more general switching sequences, non-necessarily generated by successful routes. Let ν be such a general switching sequence with length d^k on node k. For each $k = 0, 1, \ldots, K,\ l = 1, \ldots, K, K+1$ let

$$d^{k,l} = \sharp\{j : 1 \leq j \leq d^k; \nu_j^k = l\}. \tag{6}$$

We clearly have

$$d^k = \sum_{l=1}^{K+1} d^{k,l}, \tag{7}$$

for all $k = 0, \ldots, K$. However the relations

$$\sum_{l=1}^{K+1} d^{k,l} = \sum_{l=0}^{K} d^{l,k} \quad \forall k = 1, \ldots, K \tag{8}$$

and

$$\sum_{l=1}^{K+1} d^{0,l} = \sum_{l=0}^{K} d^{l,K+1} \tag{9}$$

do not hold in general.

Remark 2 *Under what conditions is a general simple switching sequence generated by a successful route? This question is clearly related to Euler graphs (we recall that a connected directed graph is an Euler graph on a set of nodes if there exists a circuit on this set of nodes using each arc of the graph exactly once), so that it is natural to call such a switching sequence a simple* Euler *switching sequence.*

The following theorem is well known (see Marshall [30] for instance): A directed graph is an Euler graph if and only if, for each node, the number of ingoing arcs is equal to the number of outgoing arcs.

A switching sequence ν on nodes $\{0, 1, \ldots, K, K+1\}$ is equivalent to the data of

- *a directed graph on this set of nodes: take the set $\{\nu^k_j,\ j = 1, \ldots, d^k\}$ as set of outgoing arcs from k,*
- *and in addition a total order on the set of arcs out of each node.*

If we just concentrate on conditions for the existence of a successful route on $\{0, 1, \ldots, K, K+1\}$ using each arc of this directed graph exactly once, the question reduces to Euler's problem by merging nodes $K+1$ and 0. Then it is immediately seen that if the switching sequences satisfy the relations (8) and (9), then the desired property holds.

However, the question whether a simple switching sequence is Euler is asking for more. Restated in graph theoretic terms, this question reads: given such a directed graph with a total order on the arcs from each node, what are the conditions ensuring the existence of a successful route on this set of nodes using each arc exactly once, and such that, for all node k, the order *in which the arcs from node k show up in this route is the same as the predefined total order on the arcs from node k?*

It should be clear that the conditions for the directed graph associated with a simple switching sequence to be Euler are necessary but in no way sufficient for the switching sequence itself to be Euler in the sense defined above.

2.2 Concatenation of Switching Sequences

Let N be a positive integer, and let $\nu(1), \ldots, \nu(N)$ be a sequence of switching sequences on $\{0, 1, \ldots, K, K+1\}$. Let $d^k(n)$ denote the length of the sequence $\nu^k(n)$. By definition, the concatenation of $\nu(1), \ldots, \nu(N)$ is the switching sequence $\nu[N] \equiv \{\nu^k[N]\}$ defined by:

$$\nu^k[N] = \{\nu^k_1(1), \ldots, \nu^k_{d^k(1)}(1), \ldots, \nu^k_1(N), \ldots, \nu^k_{d^k(N)}(N)\}, \tag{10}$$

for all $k = 0, 1, \ldots, K$, where $\nu^k[N] = \emptyset$ if $d^k(1) = \ldots = d^k(N) = 0$. The notion of concatenation will be used for other sequences later on with the same meaning.

2.3 Ordered Directed Graphs

Consider a directed graph $G = (\mathcal{N}, \mathcal{A})$, with set of nodes $\mathcal{N} = \{0, 1, 2, \ldots, K+1\}$ and with set of arcs $\mathcal{A}$. For $k = 0, 1, \ldots, K+1$, we denote I^k the set of input arcs into k and O^k the set of output arcs from k. We assume that $I^0 = O^{K+1} = \emptyset$. For $k = 0, 1, \ldots, K, K+1$ let

$$c^k = \sharp\{I^k\}, \quad d^k = \sharp\{O^k\} \tag{11}$$

and

$$d = \sum_{k=1}^{K} d^k. \tag{12}$$

Definition 3 *The directed graph G is an ordered directed graph (O.D.G.) if for each node, the output arcs are labeled in a totally ordered way.*

As already mentioned, an O.D.G. on $\mathcal{N}$ is equivalent to the data of a switching sequence ν on $\mathcal{N}$ (by mapping the j-th arc from node k to the couple (k, ν_j^k)). Thus, for an O.D.G. G, we can speak of the *path* originating from node $k = 0, \ldots, K$, or of the sequence of arcs associated with this path. If the switching sequence of the O.D.G. is generated by a route, we will also say that the O.D.G. is generated by route r.

Definition 4 *An O.D.G. G is called an Euler ordered directed graph (E.O.D.G.) if there exist an integer $N \geq 1$ and a sequence of successful routes $R = (r(1), \ldots, r(N))$, all on nodes $\{0, 1, 2, \ldots, K, K+1\}$, such that for all k, the switching sequence of this O.D.G is the concatenation $\nu^k[N]$ of N simple Euler switching sequences $\nu(1), \ldots, \nu(N)$, where $\nu(n)$ is generated by $r(n)$ for all n. In this case, we say that R is a generator of E.O.D.G. G. We will also say that the switching sequence of the O.D.G. is Euler.*

The following remarks are not difficult but they are crucial for a good understanding of the end of the section.

Remark 3 *A E.O.D.G. G may have several generators. However all generators should lead to the same variables c^k and d^k. If $R = (r(1), \ldots, r(N))$ is a generator of G, then $N = c^{K+1}$. So the number N is the same for all generators.*

Remark 4 *Let $(i_1, \ldots, i_N)$ be a permutation of $(1, \ldots, N)$. If $R = (r(1), \ldots, r(N))$ is a generator of E.O.D.G G, then it is not true in general that $R' = (r(i_1), \ldots, r(i_N))$ is also a generator of G.*

Remark 5 *If $R = (r(1), \ldots, r(N))$ is a generator of G,* route $r(1)$ *is also the* path *originating from node* 0*, and similarly, the path from node ν_1^0 is also the route $(r_2(1), r_3(1), \ldots, r_{\varphi(1)}(1))$. However, the* path *originating from node ν_n^0 is in general different from the* route $(r_2(n), r_3(n), \ldots r_{\varphi(n)}(n))$.

We now define two simple transformations of an O.D.G.

Definition 5 **(k-reduction)** *Let G be an O.D.G. on $\{0, \ldots, K+1\}$, with $\nu_1^k = l \neq K+1$. The k-reduction of G, $k \leq K$, is the O.D.G. G' on the same set of nodes, with the following characteristics:*

- *for all $p \notin \{k, l\}$, ${\nu'}^p = \nu^p$;*
- *${\nu'}^k = \{\nu_1^l, \nu_2^k, \nu_3^k, \ldots, \nu_{d^k}^k\}$, so that ${d'}^k = d^k$;*
- *${\nu'}^l = \{\nu_2^l, \nu_3^l, \ldots, \nu_{d^l}^l\}$, so that ${d'}^l = d^l - 1$ (note that we also have ${c'}^l = c^l - 1$ in view of the preceding step).*

In words, we replace the two arcs $k \to l$ and $l \to \nu_l^1$ by a single arc $k \to \nu_l^1$.

Definition 6 **(k-permutation)** *Let G be an O.D.G. on $\{0, \ldots, K+1\}$, and let σ be a permutation of $\{1, 2, \ldots, d^k\}$. The (k, σ)-permutation of G is the O.D.G. G' on the same set of nodes, with the following characteristics:*

- *for all $l \neq k$, ${\nu'}^l = \nu^l$;*
- *${\nu'}^k = \{\nu_{\sigma(1)}^k, \nu_{\sigma(2)}^k, \ldots, \nu_{\sigma(d^k)}^k\}$, so that ${d'}^k = d^k$;*

Remark 6 *Note that both k-transformations preserve the parameter d^k.*

Consider the following procedure, with input (G, k), where G is an O.D.G. and k one of its nodes:

Procedure 2 Sequential Reduction (G, k)

$G_1^k := G$; $l_1^k := k$; $t := 1$;

while $\nu_1^k(G_t^k) \cap \{1, \ldots, K\} \neq \emptyset$ **do**

begin

- $l_{t+1}^k := \nu_1^k(G_t^k)$;
- $G_{t+1}^k := k$-reduction of G_t^k
- $t := t + 1$;

end

This procedure stops after a finite number of steps, say t^*. It produces a sequence of O.D.G's $\{G_1^k, G_2^k, \ldots, G_{t^*}^K\}$ and a sequence of nodes $\{l_1^k, l_2^k, \ldots, l_{t^*}^k\}$. The O.D.G. G_t^k, $1 \leq t \leq t^*$ will be called the (k,t)-*sequential residual* of G, and $G_{t^*}^k$ will be called the k-*sequential residual* of G. The sequence of nodes $\{l_1^k, l_2^k, \ldots, l_{t^*}^k, K+1\}$ is simply the *path* from node k.

The following obvious theorem shows how to use this to reconstruct one of the generators of an E.O.D.G. from the knowledge of the associated switching sequence:

Theorem 1 *Given an E.O.D.G. G on nodes $\{0, 1, \dots, K, K+1\}$, with generator $(r(1), \dots, r(N))$, the sequential reduction of $(G, 0)$ produces a path which coincides with route $r(1)$. Let $\tilde{G}$ be the O.D.G. obtained from G by merging nodes 0 and $K+1$, then the sequential reduction of $(\tilde{G}, 0)$ produces a path which coincides with the sequence*

$$\{r_1(1), \dots, r_{\varphi(1)-1}(1), r_1(2), \dots r_{\varphi(2)-1}(2), \dots r_{\varphi(N)-1}(N)\},$$

namely the concatenation of all generator's routes in the natural order.

We conclude with two theorems on the above transformations, the first of which is obvious:

Theorem 2 *If G is an E.O.D.G. with N routes, then its 0-reduction is also an E.O.D.G with N routes.*

The proof of the next theorem is more complex and is forwarded to Appendix 8.1.

Theorem 3 *For each E.O.D.G. with N routes and for each permutation σ on $\{1, \dots, N\}$, the $(0, \sigma)$-permutation of G is also an E.O.D.G. with N routes.*

Another obvious property of Euler switching sequences is the following:

Theorem 4 *If $\nu(0)$ and $\nu(1)$ are two Euler switching sequences, so is their concatenation. The number of routes of the concatenation is the sum of the number of routes in $\nu(0)$ and $\nu(1)$.*

2.4 Parallel Reduction

The aim of this section is to investigate other routes of an E.O.D.G. than those built in Theorem 1.

Definition 7 (reduction set) *The reduction set of an O.D.G. G with respect to node k is the set of nodes which belong to ν^k and not to $\{K+1\}$:*

$$R^k(G) \equiv \{\nu_1^k, \dots, \nu_{d^k}^k\} \cap \{1, \dots, K\} \tag{13}$$

*The O.D.G. G is said to be k-*reducible *if $R^k(G) \neq \emptyset$.*

The following procedure admits as input (G, k), where G is an O.D.G. and k is one of its nodes;

Procedure 3 Parallel Reduction (G, k)

$G_1^k := G$; $X_1^k = k$; $t := 1$;

while $R^k(G_t^k) \neq \emptyset$ **do**

begin

- choose X_{t+1}^k any node in $R^k(G_t^k)$;

- let σ_t be the permutation of $\{1,\ldots,d^k\}$ such that $\nu^k_{\sigma_t(1)} = X^k_{t+1}$;
- $G'_t := \sigma_t$-permutation of G^k_t with respect to k;
- $G^k_{t+1} := k$-reduction of G'_t
- $t := t+1$;

end

Since the initial O.D.G. has a finite outdegree on each node, the procedure stops after a finite number of steps t^*, which may depend on the choices that are made. It produces

- a sequence of O.D.G.'s $\{G^k_1,\ldots,G^k_{t^*}\}$, where $G^k_1 = G$;
- a *node reduction sequence* $\{X^k_1,\ldots,X^k_{t^*}\}$, where X^k_t belongs $R^k(G^k_t) \subset \{1,2,\ldots,K+1\}$;
- a *permutation reduction sequence* $\{\sigma_1,\ldots,\sigma_{t^*}\}$.

Note that the data of the node reduction sequence is equivalent to that of the permutation reduction sequence in that the choice that is made for the permutation σ_t does not influence the (first two) output sequences of the procedure as long as $\nu^k_{\sigma_t(1)} = X^k_{t+1}$. We will call $G^k_{t^*}$ the *k-parallel residual* of K (which of course depends on the node reduction sequence X).

Remark 7 *This procedure is non-deterministic, because of the choice of reduction nodes X^k_t. Thus, it may produce a large (although finite) number of output sequences. We will denote $\mathcal{X}^k(G)$ the set of all possible sequences of reduction nodes for G.*

Remark 8 *The parallel reduction procedure admits the sequential reduction procedure as a particular case: if one take $X^k_{t+1} = \nu^k_1(G^k_t)$ for all t, then $l^k_t = X^k_t$ for all t.*

Theorem 5 *Let G be an E.O.D.G. with N routes. Then, for all parallel reductions of $(G,0)$, $t^* = d$, where d was defined in (12); for each $t < d$, G^0_t is a reducible E.O.D.G. with N routes and G^0_d is the non-reducible E.O.D.G. with N routes.*

Proof The fact that for all t, G^0_t is an E.O.D.G with N routes follows from Theorems 2-3 and an immediate induction. Let $d_t = d(G^0_t)$. A E.O.D.G. G with N routes is reducible if and only iff $d(G) > N$. Since $d_1 = d$, either $d = 0$ and the theorem is true, or $d > N$ and then, G^0_t is reducible and $d_{t+1} = d_t - 1$. An immediate induction concludes the proof. ∎

Theorem 6 *Consider an arbitrary O.D.G. If there exists a node reduction sequence X such that the 0-parallel residual of G is the non-reducible E.O.D.G with N routes, then G is an E.O.D.G. with N routes.*

Proof By assumption, there exists an integer t^*, a sequence of nodes $X = \{X_1, \ldots, X_{t^*}\}$, and a sequence of permutations $\sigma = \{\sigma_1, \ldots, \sigma_{t^*}\}$, such that when applying Procedure 3 to $(G, 0)$, the procedure stops after t^* steps, we get X as permutation node sequence (or equivalently σ as associated permutation reduction sequence), and $G^0_{t^*}$ is the non-reducible E.O.D.G. with N routes, which will be denoted G^o. Call $(0, k)$-expansion of an E.O.D.G. with N routes $(r(1), \ldots, r(N)$ the E.O.D.G. with generator $(r'(1), \ldots, r(N)$, where $r'(1) = (0, k, r_2(1), \ldots, r_{\varphi(1)}(1))$. Clearly, the $(0, k)$-expansion of an E.O.D.G. with N routes is also an E.O.D.G. with N routes. Consider the following procedure:

Procedure 4 Backward Construction (t^*, X, σ)

$t := t^*$; $H_t := G^o$;

while $t > 1$ **do**

begin

- $t := t - 1$;
- $H'_t = (0, X_{t-1})$-expansion of H_{t-1};
- $H_t = (0, \sigma_t^{-1})$-permutation of H'_t;

end

An immediate induction shows that H_t is an E.O.D.G. with N routes for all $t \leq t^*$; by construction, $H_t = G^0_t$, for all t in this range, and so $H_1 = G^0_1$ is an E.O.D.G. with N routes. ■

3 Pathwise Construction of Open Jackson-Type Queueing Networks with Finite Input

3.1 Definitions and Notations

A *network* Σ with $K \geq 1$ nodes is a quadruple

$$\Sigma = (N,\ T,\ \sigma,\ \nu)\,,$$

where $N \geq 1$ is a possibly infinite integer,

$$T = (t(1), \ldots, t(N)) \tag{14}$$

is a sequence of finite real numbers such that $t(1) \leq \ldots \leq t(N)$, which describes external arrival epochs. For each $k = 0, 1, \ldots, K$,

$$\nu = \{\nu^k_j\}_{j=1}^{d^k} \tag{15}$$

is switching sequence on $\{1, 2, \ldots, K\}$ and

$$\sigma = \{\sigma^k_j\}_{j=1}^{d^k} \tag{16}$$

is a sequence of real-valued non-negative numbers, representing service times. We assume that σ_j^k is finite for all k and $j = 1, \ldots, d^k$. Here, $d^k \in I\!N \cup \{\infty\}$ and $d^0 = N$. Station k stops serving customers once the first d^k customers have been served there.

As we will se below, these data are sufficient for the pathwise description of an open queueing network with K single-server stations, FCFS disciplines and with input sequence T, provided the rules are as follows: at time $t(1)-$, the networks is empty. External customers, numbered $n = 1, 2, \ldots, N$, arrive at epochs $t(1), \ldots, t(N)$, respectively. The n-th customer of the input is sent to station ν_n^0 (it leaves the network immediately if $\nu_n^0 = K + 1$) and is put at the end of the queue on this station. The j-th service on station k ($j = 1, 2, \ldots; k = 1, \ldots, K$) takes σ_j^k units of time. In addition, when this service is completed, the leaving customer is immediately sent to station ν_j^k (it leaves the network if $\nu_j^k = K + 1$) and it is put at the end of the queue on this station.

Remark 9 *Since we will only be interested in queue length processes and in view of our assumptions on the way services are allocated, we could replace FCFS by any non-preemptive, work-conserving discipline.*

3.2 First-Order State Variables

Consider a network Σ. Let Ψ_j^k be the epoch at which the j-th service is completed on station k. The only aim of this section is to show that each of these variables is a function of the network data (N, T, σ, ν). In the stochastic framework, this proves that the Ψ variables are indeed random variables on the probability space which carries the network data (N, T, σ, ν).

Remark 10 *The sequence $\{\Psi_j^k\}_j$ is non-decreasing, and its growth rate will be characterized by a first order ergodic theorem (a strong law of large numbers - see § 4). We will call these variables first order state variables in what follows.*

Theorem 7 *For $l = 0, \ldots, K$ and $k = 1, \ldots, K$, let $\eta^{l,k} : I\!N \to I\!N$ be the mapping*

$$\eta^{l,k}(j) = \inf\{m \geq 1 : \left[\sum_{p=1}^{m} I(\nu_p^l = k)\right] = j\}, \quad j \geq 1, \tag{17}$$

with the convention that $I(\nu_p^l = k) = 0$ for all $p \geq d^l$ and that $\eta^{l,k}(j) = \infty$ if $\sum_{p=1}^{m} I(\nu_p^l = k) < j$. In words, $\eta^{l,k}(j)$ is the smallest integer m such that the m first switching decisions out of station l produce j routings to station k. Define

$$\Psi_j^0 = \begin{cases} t(j) & \text{for } 1 \leq j \leq N; \\ \infty & \text{for } j > N, \end{cases} \tag{18}$$

and more generally, take $\Psi_j^k = \infty$ if $j > d^k$. Then the variables Ψ_j^k, $j = 1, \ldots, d^k$ can be recursively computed from the following set of evolution equations:

$$\Psi_j^k = \sigma_j^k + \max\left(\Psi_{j-1}^k, \min_{(j_0, j_1, \ldots, j_K) \in I\!N : j_0 + j_1 + \ldots + j_K = j} \left(\max_{l=0,\ldots,K} \Psi^l_{\eta^{l,k}(j_l)}\right)\right), \quad k = 1, \ldots, K, j = 1, \ldots, d^k, \tag{19}$$

with initial conditions $\Psi_0^k = -\infty$, for $k = 1, \ldots, K$.

Proof See Baccelli, Cohen and Gaujal [4]. ∎

All other variables of interest to us can be obtained from these first order state variables:

- The j-th service completion time on station k, for which the customer is sent to station l, which will be denoted $\Psi_j^{k,l}$, $k = 0, 1, \ldots, K$, $l = 1, \ldots, K, K+1$, is simply $\Psi^k_{\eta^{k,l}(j)}$.
- By assumption, $\Psi_j^k = \infty$ for $j > d^k$. By construction, we will also have $\Psi_j^k = \infty$ if $j \leq d^k$, but less than j-th customers arrive to station k, so that the j-th service is never completed by station k. So, if Φ^k denotes the total number of Ψ^k variables which are finite, then $\Phi^k \leq d^k$ for all k, and this inequality may be strict for $k = 1, \ldots, K$ (note that we nevertheless always have $\Phi^0 = d^0 = N$). We will also use $\Phi^{k,l}$ to denote the total number of variables $\Psi^{k,l}$ which are finite, and Φ, which is defined as
$$\Phi = \sum_{k=1}^{K} \Phi^k. \tag{20}$$
- Queue-length and service processes are also completely defined by the sequences $\{\Psi_j^{k,l}\}$ as we will see in § 3.8 below.

Remark 11 *We will also consider the case of* delayed networks. *A delayed network is a network to which an extra sequence of real numbers $\{\alpha_j^k\}$, $k = 1, \ldots, K$, $j = 1, \ldots, d^k$ is added (thus such a network is characterized by a 5-uple $(N, T, \sigma, \nu, \alpha)$). The rule is that the j-th service in station k cannot start before time α_j^k. The state variables $\breve{\Psi}_j^k$ of the network Σ delayed with α are defined through the recursive equation:*
$$\breve{\Psi}_j^k = \sigma_j^k + \max\left(\alpha_j^k, \breve{\Psi}_{j-1}^k, \min_{(j_0, j_1, \ldots, j_K) \in I\!N : j_0 + j_1 + \ldots + j_K = j} \left(\max_{l=0,\ldots,K} \breve{\Psi}^l_{\eta^{l,k}(j_l)}\right)\right), \tag{21}$$
with the same conventions as above.

Remark 12 *If we replace the arrival epochs $\{t(n)\}$ by $\{\hat{t}(n) \equiv t(n) + x\}$ for some fixed x then the corresponding epochs $\hat{\Psi}_j^k$ and $\hat{\Psi}_j^{k,l}$ satisfy the equations $\hat{\Psi}_j^k = \Psi_j^k + x$, $\hat{\Psi}_j^{k,l} = \Psi_j^{k,l} + x$ for all j, k, l.*

Remark 13 *Assume that $t(n-1) < t(n) = t(n+r) < t(n+r+1)$ for some $n \geq 1,\ r \geq 1,\ n+r \leq N$. If we replace the sequence $\{\nu_j^k\}$ by $\{\tilde{\nu}_j^k\}$, where*

- *$\tilde{\nu}_j^k = \nu_j^k$ for $k = 1, \ldots, K,\ j = 1, 2, \ldots$;*
- *$\tilde{\nu}_j^0 = \nu_j^0$ for $j < n$ and for $j > n + r$;*
- *$\{\tilde{\nu}_j^0,\ n \leq j \leq n+r\}$ is an arbitrary permutation of $\{\nu_j^0; n \leq j \leq n+r\}$,*

then the sequences $\{\Psi_j^{k,l}\}$ do not change (the same is true in particular for the queue-length and service processes).

3.3 Simple Euler Networks

A network is a *simple (Euler) network* if its switching sequence is generated by a successful route $r(1)$. So, for a simple network, we necessarily have $N = 1$ and $d^k = \varphi^k(1)$ for all k, where the variable $\varphi^k(1)$ is that defined in (2), for route $r(1)$. Thus the complete description of a simple network involves a real number $t(1)$ and service sequences $\{\sigma_j^k(1), 1 \le j \le \varphi^k(1)\}$. For such a network, we clearly have $\Phi^k = d^k = \varphi^k(1)$ for all k (the path from 0 is exhaustive since the switching sequence is generated by a successful route - see Remark 1).

Consider an arbitrary network with $N = d^0 = 1$. If $\Phi^k = d^k$ for all k, then this network is simple in view of Theorem 6 (we actually only need a very special case of this theorem since the parallel reduction of the O.D.G associated with ν involves no choices).

3.4 Euler Network

A network $\Sigma = (N, T, \sigma, \nu)$ is an *Euler network* if the O.D.G. associated with its switching sequence ν is an E.O.D.G. For an Euler network, there exists a sequence $R = ((r(1), \ldots, r(N))$ of successful routes which is a generator of (the O.D.G. associated with) its switching sequence ν. So $\nu = \nu[N]$, where $\nu[N]$ is the concatenation of the switching sequences $\nu(1), \ldots, \nu(N)$, and $\nu(n)$ is the simple switching sequence generated by route $r(n)$.

Theorem 8 (Conservation rule) *For an Euler network,*

$$\Phi^k = d^k \quad = \quad \varphi^k(1) + \cdots + \varphi^k(N), \tag{22}$$

for all $k = 0, \ldots, K$.

Proof Equation (22) is a direct corollary of Theorem 5. ■

So, in particular, $\Phi^{k,l}$, Φ^k and $\Phi = \Phi^1 + ...\Phi^K$ do not depend on T and σ. This result is interesting as it shows that as soon as the switching sequences of a network have the desired Euler property, then the total number of arrivals to (resp. departures from) each station of the network, as given by the recursive equation of Theorem 7, is independent of the timing information (i.e. the actual values of T and σ).

Remark 14 *If we have an infinite sequence of simple networks, say Σ_n, $n \ge 1$, we can also consider the network $\Sigma[\infty] = \Sigma_1 + \Sigma_2 + \cdots$. Let $\Sigma[N] = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_N$. It is easily checked that if $\Sigma[\infty] = (\infty, T, \sigma, \nu)$, then the queueing process (see below) in $\Sigma[N]$ coincides with that of $\Sigma'[\infty] = (\infty, T', \sigma, \nu)$, where $T' = (t(1), t(2), \ldots, t(N), \infty, \infty, \ldots)$.*

Consider a queueing network $\Sigma = (N, T, \sigma, \nu)$. The following result holds.

Theorem 9 *If $\Phi^k = d^k$ for all $k = 0, 1, \ldots, K$ then Σ is an Euler network.*

Proof This is a direct corollary of Theorem 6. ■

3.5 Composition of Networks

Consider an Euler network. Let $(r(1), \ldots, r(N))$ be a generator of its switching sequence. Let $F^k(1) = 1$ and $F^k(n+1) = F^k(n) + \varphi^k(n)$, $k = 0, 1, \ldots, K$, where $\varphi(n)$ is the parameter associated with route $r(n)$. The sequence of service times σ^k of such a network can then be seen as the concatenation $\sigma^k[N] = \{\sigma^k(1), \ldots, \sigma^k(N)\}$ of N service subsequences, where

$$\{\sigma_j^k(n)\}_{j=1}^{\varphi^k(n)} \quad = \quad \{\sigma^k_{F^k(n)}, \ldots, \sigma^k_{F^k(n+1)-1}\}.$$

In that sense, an Euler network $\Sigma = (N, T, \sigma, \nu)$ can be seen as the *composition* of N simple Euler networks $\Sigma(1), \ldots, \Sigma(N)$, where the simple network $\Sigma(n)$ is $(1, t(n), \nu(n), \sigma(n))$. We shall then write $\Sigma = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_N$.

Remark 15 *Using this terminology, we can then rephrase Theorem 9 as follows: if a network Σ is such that $\Phi^k = d^k$ for all $k = 0, 1, \ldots, K$, then there exist N simple networks $\Sigma(n)$, $n = 1, \ldots, N$, such that $\Sigma = \Sigma(1) + \Sigma(2) + \cdots + \Sigma(N)$.*

Let us make the notion of *composition of networks* more general and more precise: consider two networks $\Sigma_1 = (N_1, T_1, \sigma_1, \nu_1)$ and $\Sigma_2 = (N_2, T_2, \sigma_2, \nu_2)$, where $t_1(N_1) \leq t_1(2)$.

By definition, the composition of Σ_1 and Σ_2 is the network $\Sigma = (N, T, \sigma, \nu)$ defined by the following relations: $N = N_1 + N_2$,

$$T = (t_1(1), \ldots, t_1(N_1), t_2(1), \ldots, t_2(N_2)), \tag{23}$$

and

$$\sigma_j^k = \begin{cases} \sigma^k_{j,1}, & \text{for } 1 \leq j \leq d_1^k; \\ \sigma^k_{j-d_1^k,2}, & \text{for } j > d_1^k \end{cases}$$

and

$$\nu_j^k = \begin{cases} \nu^k_{j,1} & \text{for } 1 \leq j \leq d_1^k; \\ \nu^k_{j-d_1^k,2} & \text{for } j > d_1^k. \end{cases}$$

In general, nothing can be said on the relation between Φ_1 and Φ_2 on one side and the Φ function of the composition.

However, in the particular case where both Σ_1 and Σ_2 are (non-necessarily) simple Euler networks, then their composition is an Euler network in view of Theorem 4, so that $\Phi^k = d^k = \Phi_1^k + \Phi_2^k$ for all k. In this special case, what precedes shows that it makes sense to also note the composition of Σ_1 and Σ_2 as $\Sigma_1 + \Sigma_2$ since Σ is simply $\Sigma_1(1) + \ldots + \Sigma_1(N_1) + \Sigma_2(1) + \ldots + \Sigma_2(N_2)$. In other words, when restricted to the set of *Euler* networks, the composition rule is associative.

3.6 Monotonicity and Continuity Properties

For fixed K, N, ν and σ consider now two different input sequences: $T = \{t(n)\}_{n=1}^N$ and $\tilde{T} = \{\tilde{t}(n)\}_{n=1}^N$, and the two queueing networks: $\Sigma = (N, T, \sigma, \nu)$ and $\tilde{\Sigma} = (N, \tilde{T}, \sigma, \nu)$. The main monotonicity property is:

Theorem 10 *If $t(n) \leq \tilde{t}(n)$ for each $n = 1, \ldots, N$, then $\Psi_j^k \leq \tilde{\Psi}_j^k$ and $\Psi_j^{k,l} \leq \tilde{\Psi}_j^{k,l}$* for all j, k, l.

Proof The first proofs of this result are that of Foss [18] and Shanthikumar and Yao [33]. The proof and some extensions of this results which will be needed later on also follows from an induction argument based on the evolution equations of Theorem 7 (see Baccelli, Cohen and Gaujal [4]). ■

We now show a couple of corollaries of this result.

Corollary 1 *If $t(n) \leq \tilde{t}(n) \leq t(n) + x$ for all $n = 1, \ldots, N$, and for some $x > 0$, then*

$$\Psi_j^k \leq \tilde{\Psi}_j^k \leq \Psi_j^k + x \tag{24}$$

and

$$\Psi_j^{k,l} \leq \tilde{\Psi}_j^{k,l} \leq \Psi_j^{k,l} + x \tag{25}$$

for all j, k, l.

Proof Introduce a new network $\hat{\Sigma} = (N, \{t(n) + x\}_{n=1}^N, \ \sigma, \ \nu)$. It follows from Theorem 10 that $\Psi_j^k \leq \tilde{\Psi}_j^k \leq \hat{\Psi}_j^k$ and from Remark 12 that $\hat{\Psi}_j^k = \Psi_j^k + x$ (the same holds for $\{\Psi_j^{k,l}\}$). ■

Corollary 2 *Consider two networks: $\Sigma = (N, T, \sigma, \nu)$ and $\hat{\Sigma} = (N, T, \hat{\sigma}, \nu)$ with the same input and switching sequences but with different service times. If $\hat{\sigma}_{j_0}^{k_0} = \sigma_{j_0}^{k_0} + x$ for some $k_0 \in \{1, \ldots, K\}$ and $x > 0$, and $\hat{\sigma}_j^k = \sigma_j^k$ for all $(j, k) \neq (j_0, k_0)$, then*

$$\Psi_j^k \leq \hat{\Psi}_j^k \leq \Psi_j^k + x, \quad \forall j, k$$

(the same property holds for $\{\Psi_j^{k,l}\}$).

Proof The proof is similar to that of Corollary 1. Another simple proof can be obtained by an induction based on the equations of Theorem 7. ■

Consider now two Euler networks $\Sigma = (N, T, \sigma, \nu)$ and $\tilde{\Sigma} = (N, \tilde{T}, \tilde{\sigma}, \nu)$ with the same switching sequences, with parameters d^k, $k = 0, \ldots, K$.

Corollary 3 *If $t(n) \leq \tilde{t}(n)$ for all $n = 1, \ldots, N$ and $\sigma_j^k \leq \tilde{\sigma}_j^k$ for all $k = 1, \ldots, K, \ j = 1, \ldots, \Phi^k$, then*

$$\Psi_j^k \leq \tilde{\Psi}_j^k \leq \Psi_j^k + \max_{1 \leq n \leq N} (\tilde{t}(n) - t(n)) + \sum_{l=1}^{K} \sum_{i=1}^{d^l} (\tilde{\sigma}_i^l - \sigma_i^l) \tag{26}$$

for all j, k (the same holds true for $\Psi_j^{k,l}$).

Proof This result follows immediately from Corollaries 1-2 and from induction arguments. ■

Remark 16 *(Continuation of Remark 11) It is easy to check that if* $\alpha_j^k(1) \leq \alpha_j^k(2)$ *for all* j *and* k*, then the network* Σ*, when delayed with* $\alpha(1)$ *and* $\alpha(2)$ *respectively, leads to state variables that satisfy the relation*

$$\breve{\Psi}_j^k(1) \leq \breve{\Psi}_j^k(2), \quad \forall j, k.$$

In particular, a delayed network is always a majorant of the non-delayed network in the sense mentioned above.

Fix now K, N and an Euler switching sequence with N routes ν, and consider a set of sequences $\{t_\epsilon(n)\}_{n=1}^N$ and $\{\sigma_{j,\epsilon}^k\}_{j=1}^{d^k}$, for $k = 1, \ldots, K$, where $\epsilon > 0$.

Corollary 4 (Continuity property) *Assume that*

$$t_\epsilon(n) \to t(n),\ \sigma_{j,\epsilon}^k \to \sigma_j^k \tag{27}$$

as $\epsilon \to 0$ *for all* $n = 1, \ldots, N,\ k = 1, \ldots, K,\ j = 1, \ldots, d^k$*. Then*

$$\Psi_{j,\epsilon}^k \to \Psi_j^k \tag{28}$$

for each $k = 1, \ldots, K,\ j = 1, \ldots, d^k$ *(the same holds true for* $\Psi_{j,\epsilon}^{k,l}$*).*

Proof The proof follows immediately from Corollary 3. ■

Corollary 5 *Let* Σ *be the composition of the Euler networks* Σ_1 *and* Σ_2*. Then*

$$\Psi_{j+d_1^k}^k \geq \Psi_{j,2}^k \tag{29}$$

and

$$\Psi_i^k \leq \Psi_{i,1}^k \tag{30}$$

for each $k = 1, \ldots, K,\ j = 1, 2, \ldots,\ i = 1, \ldots, d^k$ *(the same holds for* $\Psi_j^{k,l}$*).*

Proof We prove (29) only (the proof of (30) is similar). We construct an auxiliary network $\tilde{\Sigma}$ with driving sequences $(\{\tilde{t}(n)\}_{n=1}^N,\ \{\tilde{\sigma}_j^k\},\ \{\tilde{\nu}_j^k\})$ obtained by shifting the sequence T of Σ to the left in such a way that the two networks Σ_1 and Σ_2 *separate*, namely the last departure from the customers of the first network takes place before the first arrival of the second network of the composition. More precisely let

$$\Delta = \max_{0 \leq l \leq N_1} \{\Psi_{d_1^l,1}^l - t_1(1)\}. \tag{31}$$

We take

- $\tilde{\sigma}_j^k = \sigma_j^k$ and $\tilde{\nu}_j^k = \nu_j^k$ for all j, k;
- $\tilde{t}(2) = \min\{t_1(1), t_2(1) - \Delta\}$, $\tilde{t}(n+1) = \tilde{t}(n) + t_1(n+1) - t_1(n)$, for $n \leq N_1$ and $\tilde{t}(n) = t(n)$ for $n > N_1$.

Since $\tilde{t}(n) \le t(n)$ for all $n \ge 1$, Theorem 10 implies that

$$\tilde{\Psi}_j^k \le \Psi_j^k \tag{32}$$

for all j, k. But since the last customer of $\tilde{\Sigma}_1$ leaves the network before the arrival of the first customer of Σ_2 (it is in that sense that the networks are separated), then

$$\tilde{\Psi}_{j+d_1^k}^k = \Psi_{j,2}^k$$

for all j, k. ■

Remark 17 *The notion of separation of the composition of two networks which is introduced in the proof of the preceding corollary is quite crucial and will be used at several occasions later on.*

3.7 The Space D_+^0

Let $f : [0, \infty) \to \{0, 1, 2, \ldots\}$ be a right-continuous non-increasing function with compact support, i.e.

$$b(f) \equiv \sup\{x : f(x) > 0\} < \infty \tag{33}$$

and $D_+^0 \equiv D_+^0[0, \infty)$ be the space of such functions. For $f \in D_+^0$, let $a(f) = f(0)$.

As we shall see in the next section, this space contains the second-order variables associated with a network. We show below that D_+^0 is actually a separable metric space, endowed with a natural partial order.

Let H be the set of continuous and strictly increasing functions $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0,\ h(\infty) = \infty$. For $f, g \in D_+^0$, consider the (Skorohod) distance

$$d(f, g) = \inf_{h \in H} \{\sup_{x \ge 0} |h(x) - x| + \sup_{x > 0} |f(h(x)) - g(x)|\}. \tag{34}$$

The space (D_+^0, d) is separable (see Gihman-Skorohod [23], Chapter 9, §5) and possesses the following properties:

- It admits the partial order $\le$ defined by $f \le g$ if $f(x) \le g(x)$ for all $x \ge 0$.
- If the sequence $\{f_n\}$, $f_n \in D_+^0$ is Cauchy (w.r. to d), then there exists a function $g \in D_+^0$ such that $g \ge f_n$, for all $n \ge 0$.
- If the sequence $\{f_n\}$, $f_n \in D_+^0$ is monotone increasing (non-decreasing) and if $\lim_n a(f_n) < \infty$ and $\lim_n b(f_n) < \infty$, then $\lim_n f_n \equiv f$ belongs to D_+^0, and $d(f_n, f) \to 0$.

Remark 18 *Let m and k be fixed; for each pair of functions F^1 and F^2 of the form: $F^l = \sum_{i=1}^m f_i^l - \sum_{j=1}^k g_j^l$, where all f_i^l and g_j^l belong to D_+^0, $l = 1, 2$, let*

$$d(F^1, F^2) = \sum_{i=1}^m d(f_i^1, f_i^2) + \sum_{j=1}^k d(g_j^1, g_j^2).$$

If $f_i^n \in D_+^0$ converges monotonically to $f_i \in D_+^0$ for each $i = 1, \ldots, m$, and $g_j^n \in D_+^0$ converges monotonically to $g_j \in D_+^0$ for each $j = 1, \ldots, k$, then the functions $F^n \equiv \sum_{i=1}^m f_i^n - \sum_{j=1}^k g_j^n$ converge to the function $F \equiv \sum_{i=1}^m f_i - \sum_{j=1}^k g_j$ pointwise and with respect to distance d.

3.8 Second-Order State Variables

Consider an Euler network with parameter N. For each k, l, consider the processes

$$\begin{aligned}
\bar{\Gamma}^{k,l}(t) &= \Phi^{k,l} - \sum_{j=1}^{\Phi^{k,l}} I(\Psi_j^{k,l} \leq t) && (35)\\
\bar{\Gamma}^{k}(t) &\equiv \sum_{l=1}^{K+1} \bar{\Gamma}^{k,l}(t) = \Phi^{k} - \sum_{j=1}^{\Phi^{k}} I(\Psi_j^{k} \leq t) && (36)\\
\bar{\Gamma}(t) &\equiv \sum_{k=1}^{K} \bar{\Gamma}^{k}(t), && (37)
\end{aligned}$$

(where $\Phi^0 = N$ and $\Psi_j^0 = t(j)$), which count the number of departures from station k to station l (resp. from station k or from all stations) taking place after time t.

The processes $\bar{\Gamma}^{k,l}(t)$ and $\bar{\Gamma}^{k}(t)$ are right-continuous and belong to D_+^0.

We will also need the following second-order processes:

- $\bar{Q}^k(t)$ is the queue-length on station k at time t (including the customer in service);
- $\bar{\chi}^k(t)$ is the residual service time of the customer in service at time $t+$ in station k (0 if $\bar{Q}^k(t) = 0$).

These processes are defined from the $\bar{\Gamma}$ functions through the following relations:

$$\begin{aligned}
\bar{Q}^k(t) &= \bar{\Gamma}^{k}(t) - \sum_{l=0}^{K} \bar{\Gamma}^{l,k}(t); && (38)\\
\bar{Q}(t) &\equiv \sum_{k=1}^{K} \bar{Q}^k(t) = \sum_{k=1}^{K} \bar{\Gamma}^{k}(t) - \sum_{k=1}^{K}\sum_{l=0}^{K} \bar{\Gamma}^{l,k}(t) \\
&= \sum_{k=1}^{K} \bar{\Gamma}^{k,K+1}(t) - \sum_{k=1}^{K} \bar{\Gamma}^{0,k}(t); && (39)\\
\bar{\chi}^k(t) &= \inf\{v > t : \bar{\Gamma}^{k}(v) < \bar{\Gamma}^{k}(t)\} - t, && (40)
\end{aligned}$$

where the last relation assumes that $\bar{Q}^k(t) > 0$. We call these variables second-order variables because they are defines as differences (of counting measures) of first order ones. ¿From Theorem 10 and its corollaries, we get:

Lemma 1 *Consider two networks:* $\Sigma_1 = (N, T_1, \sigma_1, \nu_1\})$ *and* $\Sigma_2 = (N, T_2, \sigma_2, \nu_2\})$. *If* $t_1(n) \leq t_2(n)$, $\sigma_{j,1}^k \leq \sigma_{j,2}^k$ *and* $\nu_{j,1}^k = \nu_{j,2}^k$, *for all* $n = 1, \ldots, N,\ k = 1, \ldots, K,\ j = 1, 2, \ldots$, *then*

$$\bar{\Gamma}_1^{k,l}(t) \leq \bar{\Gamma}_2^{k,l}(t) \qquad (41)$$

for all k, l *and for all* $-\infty < t < \infty$.

We will also need the functions describing the *residual* departure processes and the residual queue length processes. Let

$$\Gamma^{k,l}(t) = \bar{\Gamma}^{k,l}(t + t(N)), \quad t \geq 0 \tag{42}$$

and

$$\Gamma^k(t) = \bar{\Gamma}^k(t + t(N)), \quad t \geq 0. \tag{43}$$

Note that

$$\Gamma^k(t) = \sum_{l=1}^{K+1} \Gamma^{k,l}(t), \quad t \geq 0. \tag{44}$$

Remark 19 *The processes $\Gamma^{k,l}(t)$ do not depend on the values $t(1), \ldots, t(N)$ but only on their increments $t(n+1) - t(n)$, $n = 1, \ldots, N-1$. This means, in particular, that if we consider two networks Σ and $\hat{\Sigma}$ with the same service times and switching decisions and with inputs $\{t(n)\}$ and $\{\hat{t}(n)\}$ satisfying the equations $\hat{t}(n) = t(n) + C$ for some $C \geq 0$ and for all n, then $\Gamma^{k,l}(t) = \hat{\Gamma}^{k,l}(t)$ for all t, k, l (the same is true for $\Gamma^k(t)$).*

Let $\tau(n) = t(n+1) - t(n), n = 1, \ldots, N-1$.

Lemma 2 (Monotonicity property) *Consider two networks $\Sigma_1 = (N, T_1, \sigma_1, \nu_1)$ and $\Sigma_2 = (N, T_2, \sigma_2, \nu_2)$. If $\tau_1(n) \geq \tau_2(n)$, $\sigma^k_{j,1} \leq \sigma^k_{j,2}$ and $\nu^k_{j,1} = \nu^k_{j,2}$ for all $n = 1, \ldots, N-1$, $k = 1, \ldots, K$, $j = 1, 2, \ldots$, then*

$$\Gamma_1^{k,l}(t) \leq \Gamma_2^{k,l}(t) \tag{45}$$

for all k, l, t.

Proof The processes to be compared involve different epochs: $t+t_1(N)$ and $t+t_2(N)$, respectively. For connecting these two epochs, introduce two new networks:

$$\tilde{\Sigma}_1 = (N, \tilde{t}_1, \{\sigma^k_{j,1}\}, \{\nu^k_{j,2}\}) \tag{46}$$

and

$$\tilde{\Sigma}_2 = (N, \tilde{t}_2, \{\sigma^k_{j,2}\}, \{\nu^k_{j,2}\}), \tag{47}$$

where $\tilde{t}_1(n) = C - \sum_{j=n}^{N-1} \tau_1(j)$, for $n < N$, $\tilde{t}_1(N) = C$, $\tilde{t}_2(n) = C - \sum_{j=n}^{N-1} \tau_2(j)$, for $n < N$, $\tilde{t}_2(N) = C$ and $C = \max(t_1(N), t_2(N))$.

¿From Remark 19,

$$\tilde{\Gamma}_1^{k,l}(t) = \Gamma_1^{k,l}(t) \quad \text{and} \quad \tilde{\Gamma}_2^{k,l}(t) = \Gamma_2^{k,l}(t), \tag{48}$$

for all k, l, t. Since $\tilde{t}_1(n) \leq \tilde{t}_2(n)$ for each n, then $\tilde{\Gamma}_1^{k,l}(t) \leq \tilde{\Gamma}_2^{k,l}(t)$ for all k, l, t. ■

Similarly, the *residual* queue-length processes and the residual service-time processes are defined by the relations:

$$Q^k(t) = \bar{Q}^k(t + t(N)), \quad Q(t) = \bar{Q}(t + t(N)), \tag{49}$$

$$\chi^k(t) = \bar{\chi}^k(t + t(N)), \quad t \geq 0. \tag{50}$$

We have

$$Q^k(t) = \Gamma^k(t) - \sum_{l=1}^{K} \Gamma^{l,k}(t) \equiv \sum_{i=1}^{K+1} \Gamma^{k,i}(t) - \sum_{l=1}^{K} \Gamma^{l,k}(t), \tag{51}$$

$$\chi^k(t) = \inf\{v > t : \Gamma^k(v) < \Gamma^k(t)\} - t \tag{52}$$

if $Q^k(t) > 0$ ($\chi^k(t) = 0$ if $Q^k(t) = 0$), and

$$Q(t) = \sum_{k=1}^{K} \Gamma^{k,K+1}(t) \tag{53}$$

for $k = 1, \ldots, K,\ t \geq 0$. This last formula gives the following corollary:

Corollary 6 *Under the conditions of Lemma 2,*

$$Q_1(t) \leq Q_2(t) \tag{54}$$

for all $t \geq 0$.

Returning now to the composition of networks (see § 3.5), we can formulate the following immediate corollary of Lemma 2:

Corollary 7 *If the network Σ is the composition of two Euler networks Σ_1 and Σ_2, then*

$$\Gamma^{k,l}(t) \geq \Gamma_2^{k,l}(t), \quad Q(t) \geq Q_2(t) \tag{55}$$

for all k, l, t.

Associated with any network Σ, we introduce the new variable:

$$Z = \inf\{t \geq 0 : \max_{1 \leq k \leq K} \Gamma^k(t) = 0\}, \tag{56}$$

which represents the time to empty the system, measured from the last external arrival.

Lemma 3 *If Σ is the composition of the Euler networks Σ_1 and Σ_2, then*

$$(Z - x - y)^+ \leq (Z_1 - x)^+ + (Z_2 - y)^+, \tag{57}$$

for all $x \geq 0,\ y \geq 0$.

Proof It is enough to consider the case $x = y = 0$ only. If $z \equiv t_2(1) - t_1(N_0) \geq Z_1$, then the two networks are separated and $Z = Z_2 \leq Z_1 + Z_2$. If $z < Z_1$, let $\hat{\Sigma}$ be the composition of the networks Σ_1 and $\hat{\Sigma}_2$, where

$$\hat{\Sigma}_2 = (\{\hat{t}_2(n)\}_{n=1}^{N_2},\ \{\sigma_{j,2}^k\},\ \{\nu_{j,2}^k\})$$

with $\hat{t}_2(n) = t_2(n) + (Z_1 - z),\ n = 1, \ldots, N_2$. By construction, $\hat{Z} = Z_2$. Lemma 1 implies that $Z + t_2(N_2) \leq \hat{Z} + t_2(N_2) + Z_1 - z$. So $Z \leq Z_2 + Z_1 - z \leq Z_2 + Z_1$ (z is non-negative by definition). ■

Remark 20 *The same monotonicity and sub-additive properties hold true for networks with multi-server stations (with FCFS disciplines), provided we still associate service times and switching decisions with stations. More precisely, we have to assume that, on each station k, the j-th service takes σ_j^k units of time (regardless of the server to which the customer is allocated), and that after this service, the customer is sent to station ν_j^k (see Shanthikumar and Yao [33] for the monotonicity property).*

4 First-Order Ergodic Properties

4.1 Basic Definitions and Notations

Consider a sequence of simple Euler networks, say $\{\Sigma(n)\}_{n=-\infty}^{\infty}$, where $\Sigma(n) = (1, t(n), \sigma(n), \nu(n))$ and where the switching decision sequence $\nu(n)$ is that generated by the route $r(n) = (r_1(n), r_2(n), \ldots, r_{\phi(n)}(n))$. We assume that $t(n) \leq t(n+1)$ for all n and we denote $\tau(n)$ the difference $t(n+1) - t(n)$. Associated with the sequence $\{\Sigma(n)\}$, we define the following basic sequences $u(n)$ and $\{S^k(n)\}$

- $u(0) = 0$ and $u(n+1) - u(n) = \tau(n)$ for all n;
- $S^k(n) = \sum_{j=1}^{d^k(n)} \sigma_j^k(n)$ and $S(n) = S^1(n) + \cdots + S^K(n)$ for all $-\infty < n < \infty,\ k = 1, \ldots, K$;

Similarly, we define $\{\sigma_j^k\}$ and $\{\nu_j^k\}$ to be the following infinite concatenation of the $\{\sigma_j^k(n)\}$ and $\{\nu_j^k(n)\}$ sequences:

- Let $F^k(n) = \varphi^k(1) + \cdots + \varphi^k(n)$ for $n \geq 1$ and $F^k(n) = \varphi^k(n) + \cdots + \varphi^k(0)$ for $n \leq 0$;
- $\nu_j^0 = r_1(j)$, for $-\infty < j < \infty$;
- for $k = 1, \ldots, K$,
  - for $0 < j \leq \varphi^k(1)$, $\sigma_j^k = \sigma_j^k(1)$ and $\nu_j^k = \nu_j^k(1)$;
  - for $n \geq 1$, $F^k(n) < j \leq F^k(n+1)$, $\sigma_j^k = \sigma_{j-F^k(n)}^k(n+1)$ and $\nu_j^k = \nu_{j-F^k(n)}^k(n+1)$;
  - for $-\varphi^k(0) < j \leq 0$ $\sigma_j^k = \sigma_{j+\varphi^k(0)}^k(0)$ and $\nu_j^k = \nu_{j+\varphi^k(0)}^k(0)$;
  - for $n \geq 0$, $-F^k(-n-1) < j \leq -F^k(-n)$, $\sigma_j^k = \sigma_{j+F^k(-n-1)}^k(-n-1)$ and $\nu_j^k = \nu_{j+F^k(-n-1)}^k(-n-1)$.

Assume that we have a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, endowed with an ergodic measure-preserving shift θ. The symbols $\theta^n, n \geq 0$, will denote the iterations of this transformation (so that $\theta^1 = \theta$, while θ^0 is the identity), and the symbol θ^{-n} stands for the transformation inverse to θ^n, $n = 1, 2, \ldots$. The same symbol θ will also be used for the measure-preserving shift on the events of $\mathcal{F}$. Let

$$\xi(n) = \{\tau(n),\ \{\sigma(n)\},\ \{\nu(n)\}\}. \tag{58}$$

Our stochastic assumptions will be as follows:

- the variables $t(n),\ \{\sigma(n)\},\ \{\nu(n)\}$ are random variables defined on $(\Omega, \mathcal{F}, P)$;
- the random variables $\xi(n)$ satisfy the relation $\xi(n) = \xi(0) \circ \theta^n$ for all n, which implies that $\{\xi(n)\}_{n=-\infty}^{\infty}$ is stationary and ergodic;
- all the expectations $\mathbf{E}\varphi^k(0),\ \mathbf{E}S^k(0) = b^k,\ \mathbf{E}\tau(0) = \lambda^{-1}$ are finite.

Without loss of generality, we can assume that $\mathbf{E}S^k(0) > 0$ for all k.

For $m \le n$ let

$$\Sigma_{[m,n]} = \Sigma(m) + \cdots + \Sigma(n),$$

where + is the composition rule introduced in § 3.4. We have in particular $\Sigma_{[n,n]} = \Sigma(n)$. The composition assumption implies that for each $m < l \le n$,

$$\Sigma_{[m,n]} = \Sigma_{[m,l-1]} + \Sigma_{[l,n]}.$$

Let $X_{[m,n]}$ be the time to empty the system measured from time $t(0)$:

$$X_{[m,n]} = t(n) - t(0) + Z_{[m,n]}, \tag{59}$$

where $Z_{[m,n]}$ represents the variable defined in (56) for the network $\Sigma_{[m,n]}$, for $-\infty < m \le n < \infty$. We shall also use the notation

$$X_n = X_{[0,n]}. \tag{60}$$

4.2 First-Order Ergodic Theorem

The variable X_n, which can be seen as the maximum over all j and k of the Ψ_j^k variables in network $\Sigma_{[0,n]}$ measured from $t(0)$ (and equivalently the variables $Z_{[0,n]}$ or $Z_{[-n,0]}$) satisfy a SLLN:

Theorem 11 *Under the above conditions, there exists a finite non-negative constant γ such that*

$$\lim \frac{Z_{[-n,0]}}{n} = \lim \frac{Z_{[-n,-1]}}{n} = \lim \frac{\mathbf{E}Z_{[-n,0]}}{n} = \lim \frac{\mathbf{E}Z_{[-n,-1]}}{n} = \gamma \tag{61}$$

a.s. as $n \to \infty$.

Proof It follows from Lemma 3 that

$$Z_{[-n,-1]} \le Z_{[-n,-l-1]} + Z_{[-l,-1]} \tag{62}$$

for all $1 \le l < n$. Since $Z_{[-n,-l-1]} = Z_{[-n+l,-1]} \circ \theta^{-l}$ and $0 \le \mathbf{E}Z_{[0,0]} \le \mathbf{E}S(0) < \infty$, Kingman's subadditive ergodic theorem allows us to complete the proof. ■

Corollary 8 *Under the above conditions*

$$\lim \frac{Z_{[1,n]}}{n} = \lim \frac{\mathbf{E}Z_{[1,n]}}{n} = \gamma \tag{63}$$

and

$$\lim \frac{X_n}{n} = \lim \frac{\mathbf{E}X_n}{n} = \gamma + \lambda^{-1} \tag{64}$$

a.s. as n tends to ∞.

Remark 21 *Consider the more general situation when the sequence $\{\xi(n)\}_{n=0}^{\infty}$ couples with a stationary sequence. If the stationary sequence under consideration satisfies the above assumptions, then*

$$\lim \frac{Z_{[1,n]}}{n} = \gamma, \quad \lim \frac{X_n}{n} = \gamma + \lambda^{-1} \tag{65}$$

a.s. as $n \to \infty$. If, in addition, all the expectations $\mathbf{E}\varphi^k(n)$, $\mathbf{E}S^k(n)$, $\mathbf{E}\tau(n)$ are finite and the coupling time is integrable, then the statement of Corollary 8 is still true.

4.3 Finiteness of Second-Order Variables

The monotonicity property of Corollary (7) implies that $Z_{[-n-1,0]} \geq Z_{[-n,0]}$ a.s. for all non-negative n. So there exists an a.s. limit $\lim Z_{[-n,0]}$ as n tends to ∞ (which may be either finite or infinite).

In relation with the network $\Sigma_{[-n,0]}$ and for $k, 1, \ldots, K$, and $t \geq 0$, we also introduce the processes $\Gamma^{k,l}_{[-n,0]}(t)$, $\Gamma^{k}_{[-n,0]}(t)$, $\Gamma_{[-n,0]}(t)$, $Q^{k}_{[-n,0]}(t)$ and $Q_{[-n,0]}(t)$, which are defined as in § 3.8. Let

$$\Gamma^{k}_{[-n,0]} = \Gamma^{k}_{[-n,0]}(0); \tag{66}$$

and

$$D^{k}_{[-n,0]} = \sum_{j=0}^{\Gamma^{k}_{[-n,0]}-1} \sigma^{k}_{-j}, \quad E^{k}_{[-n,0]} = \sum_{j=0}^{\Gamma^{k}_{[-n,0]}-2} \sigma^{k}_{-j} \tag{67}$$

(here $\sum_0^{-1} = \sum_0^{-2} \equiv 0$). The monotonicity property also implies that $\Gamma^{k,l}_{[-n,0]}(t)$, $\Gamma^{k}_{[-n,0]}(t)$, $Q_{[-n,0]}(t)$, $\Gamma^{k}_{[-n,0]}$, $D^{k}_{[-n,0]}$ and $E^{k}_{[-n,0]}$ are non-decreasing in n. It follows from the definitions that

$$\max_{1\leq k\leq K} E^{k}_{[-n,0]} \leq Z_{[-n,0]} \leq \sum_{k=1}^{K} D^{k}_{[-n,0]} \tag{68}$$

for all n. So $Z_{[-n,0]} \to \infty$ as $n \to \infty$ iff there exists a.s. $k \in \{1, \ldots, K\}$ such that $\Gamma^{k}_{[-n,0]} \to \infty$ as $n \to \infty$ (this is true because we assume $\mathbf{E}S^k(0)$ to be positive for all k).

Let A be the event

$$A = \{\lim_{n\to\infty} Z_{[-n,0]} = \infty\}. \tag{69}$$

Theorem 12 *Under the conditions of § 4.1, either* $\mathbf{P}(A) = 1$ *or* $\mathbf{P}(A) = 0$.

Proof We shall prove that a.s. if $Z_{[-n,0]} \to \infty$ then $Z_{[-n,1]} \to \infty$ as n tends to ∞ . If it is so, then $\theta A \supseteq A$. But the shift θ is measure-preserving, so $\mathbf{P}(\theta A - A) = 0$. Since θ is ergodic, the last equality implies $\mathbf{P}(A) \in \{0, 1\}$.

It follows from (68) that it is enough to prove that $\mathbf{P}(\Gamma^{k}_{[-n,0]} \to \infty) \in \{0, 1\}$. For this it is sufficient to show that a.s. if $\Gamma^{k}_{[-n,0]} \to \infty$, then $\Gamma^{k}_{[-n,1]} \to \infty$.

For each $N >> 1$ we can a.s. choose $l \equiv l_N$ such that $\sum_{j=N+1}^{N+l} \sigma^{k}_{-j} \geq \tau(0)$. Since $\Gamma^{k}_{[-n,0]} \to \infty$, there exists n_N such that $\Gamma^{k}_{[-n,0]} > N + l$ for all $n \geq n_N$. Therefore $\Gamma^{k}_{[-n,1]} \geq N$ for all $n \geq n_N$. ■

Corollary 9 *If $P(A) = 0$ then the random variables $Z_{[-n,0]}$ converge monotonically a.s. to a finite random variable $Z(0)$, and if we define*

$$Z(m) = Z(0) \circ \theta^m$$

then

$$Z_{[-n+m,m]} \equiv Z_{[-n,0]} \circ \theta^m \leq Z(m) \quad a.s. \tag{70}$$

for all $0 \leq m, n < \infty$.

It follows from Theorem 12 and from its proof that similar results hold true for the processes $\Gamma^{k,l}_{[-n,0]}(t),\ \Gamma^{k}_{[-n,0]}(t),\ \Gamma_{[-n,0]}(t)$ and $Q_{[-n,0]}(t)$:

Corollary 10 *If $\mathbf{P}(A) = 0$ then the processes $\Gamma^{k,l}_{[-n,0]}(t),\ \Gamma^{k}_{[-n,0]}(t),\ \Gamma_{[-n,0]}(t)$ and $Q_{[-n,0]}(t)$ converge monotonically a.s. to finite processes $\Gamma^{k,l}(t) \in D_0^+,\ \Gamma^k(t) \in D_0^+,\ \Gamma(t) \in D_0^+$ and $Q(t) \in D_0^+$, respectively.*

Denote by $Q^k_{[-n,0]} \equiv Q^k_{[-n,0]}(0)$ the queue-length and by $\chi^k_{[-n,0]} \equiv \chi^k_{[-n,0]}(0)$ the residual service time on station k in the network $\Sigma_{[-n,0]}$ at time $t(0)$ (where $\chi^k_{[-n,0]} = 0$ if $Q^k_{[-n,0]} = 0$).

Corollary 11 *If $\mathbf{P}(A) = 0$, then the r.v.'s $Q^k_{[-n,0]}$ and $\chi^k_{[-n,0]}$ converge weakly to some a.s. finite r.v.'s Q^k and χ^k, respectively, as $n \to \infty$.*

More results are available on these first-order ergodic theorems. One of the most interesting is the solidarity property of Corollary 22, Appendix C.

4.4 Scaling

In what follows, it will be useful to consider various scalings of the arrival processes: for each scaling factor $0 \leq C < \infty,\ -\infty < m \leq n < \infty$, consider the sequences

$$\xi(n, C) = \{C\tau(n),\ \{\sigma^k_j(n)\},\ \{\nu^k_j(n)\}\}, \tag{71}$$

the simple networks

$$\Sigma(m, C) = \{1, Ct(n),\ \{\sigma^k_j(n)\},\ \{\nu^k_j(n)\}\} \tag{72}$$

and the Euler networks $\Sigma_{[m,n]}(C) = \Sigma(m, C) + \cdots + \Sigma(n, C)$. Let

$$\gamma(C) = \lim_{n\to\infty} \frac{Z_{[-n,0]}(C)}{n} \tag{73}$$

(here $\gamma(1) = \gamma$). For instance, for a $G/G/1$ queue (with obvious notations)

$$\gamma(C) = \left(\frac{1}{\mu} - \frac{C}{\lambda}\right)^+.$$

Lemma 4 *$\gamma(C)$ is a continuous and non-increasing function.*

Proof For each $n \geq 0,\ C \geq 0,\ \epsilon \geq 0$

$$Z_{[-n,0]}(C + \epsilon) \leq Z_{[-n,0]}(C) \leq Z_{[-n,0]}(C + \epsilon) + \epsilon(-t(-n)), \quad a.s.$$

So

$$\gamma(C + \epsilon) \leq \gamma(C) \leq \gamma(C + \epsilon) + \epsilon\lambda^{-1}.$$

■

5 Stability Conditions

5.1 Main Ergodic Theorems

For $-\infty < m \leq n < \infty$, let

$$Y_{[m,n]} = Z_{[m,n]}(0) \tag{74}$$

and

$$\rho = \lambda\gamma(0), \tag{75}$$

where the notations are those of the end of § 4.3. The random variable $Y_{[m,n]}$ simply represents the time to empty the system, measured from time $t(m)$ on, when a bulk of $m - n$ customers arrives at time $t(m)$. We know that

$$\begin{aligned}
\gamma(0) &= \lim_{n\to\infty} \frac{Y_{[-n,0]}}{n} = \lim_{n\to\infty} \frac{Y_{[-n,-1]}}{n} \quad a.s. \\
&= \lim_{n\to\infty} \frac{\mathbf{E}Y_{[-n,0]}}{n} = \lim_{n\to\infty} \frac{\mathbf{E}Y_{[-n,-1]}}{n}.
\end{aligned}$$

Theorem 13 *If $\rho < 1$, then $\mathbf{P}(A) = 0$.*

Proof For each $l \geq 0$, let N_l be the random variable

$$N_l = \min\{n \geq 0 : Z_{[-n,0]} \geq u(l)\}. \tag{76}$$

If $\mathbf{P}(A) = 1$ then $N_l < \infty$ a.s. for each l. By definition (see (59)), for all $n, l \geq 0$, the equalities

$$X_{[-n,l]} = t(l) - t(0) + Z_{[-n,l]} = u(l) + Z_{[0,n+l]} \circ \theta^{-n} \tag{77}$$

hold. Consider the network $\hat{\Sigma}_{[-n,l]}$ obtained by composing the simple networks $\hat{\Sigma}(-n)$,. . . , $\hat{\Sigma}(l)$, where $\hat{\Sigma}(j)$ has the same service times and switching decisions as $\Sigma(j)$ but an arrival epoch $\hat{t}(j)$ defined as follows:

$$\hat{t}(j) = \begin{cases} t(j) & \text{for } -n \leq j \leq 0, \\ Z_{[-n,0]} & \text{for } 1 \leq j \leq l, \end{cases} \tag{78}$$

For $n \geq N_l$, $t(j) \leq \hat{t}(j)$ for all $-n \leq j \leq l$, so that

$$u(l) + Z_{[-n,l]} \leq Z_{[-n,0]} + Y_{[1,l]},$$

as a direct consequence of the monotonicity property of Lemma 2. Therefore, if $n \geq N_l$,

$$u(l) + Z_{[-n-l,0]} \circ \theta^l \leq Z_{[-n,0]} + Y_{[-l+1,0]} \circ \theta^l, \tag{79}$$

Consider now the network $\tilde{\Sigma}_{[-n,l]}$ defined as above with

$$\tilde{t}(j) = \begin{cases} t(j) & \text{for } -n \leq j \leq 0, \\ t(j) + Z_{[-n,0]} + S_{[1,j-1]} & \text{for } 1 \leq j \leq l, \end{cases} \tag{80}$$

(where $\sum_{i=1}^{0} = 0$). By definition, in network $\tilde{\Sigma}_{[-n,l]}$, all external arrivals taking place later than (and including at) $\tilde{t}(1)$ find an empty system. So, in particular,

$$\tilde{Z}_{[-n,l]} \leq S(l). \tag{81}$$

The monotonicity property of Lemma 2 also implies that

$$\begin{aligned} X_{[-n,l]} &= u(l) + Z_{[-n-l,0]} \circ \theta^l \\ &\leq \tilde{X}_{[-n,l]} = \tilde{u}(l) + \tilde{Z}_{[-n,l]} \\ &= Z_{[-n,0]} + u(l) + S_{[1,l]}. \end{aligned}$$

So

$$\theta^l Z_{[-n-l,0]} - Z_{[-n,0]} \leq S_{[1,l]} \tag{82}$$

for all $n, l \geq 1$. Combining (82) and (79), we get the inequalities

$$\theta^l Z_{[-n-l,0]} - Z_{[-n,0]} \leq (\theta^l Y_{[-l+1,0]} - u(l)) I(n \geq N_l) + S_{[1,l]} I(n < N_l), \quad a.s. \tag{83}$$

All of random variables in the last inequality are integrable. So

$$\mathbf{E}(\theta^l Z_{[-n-l,0]} - Z_{[-n,0]}) \leq \mathbf{E}\{(\theta^l Y_{[-l+1,0]} - u(l)) I(n \geq N_l)\} + \mathbf{E}\{S_{[1,l]} I(n < N_l)\}.$$

But

$$\begin{aligned} \mathbf{E}(\theta^l Z_{[-n-l,0]} - Z_{[-n,0]}) &= \mathbf{E}(\theta^l Z_{[-n-l,0]}) - \mathbf{E}(Z_{[-n,0]}) \\ &= \mathbf{E}(Z_{[-n-l,0]}) - \mathbf{E}(Z_{[-n,0]}) = \mathbf{E}(Z_{[-n-l,0]} - Z_{[-n,0]}) \geq 0, \end{aligned}$$

because $Z_{[-n,0]}$ is a non-decreasing sequence. We have

$$\begin{aligned} \mathbf{E}\{(\theta^l Y_{[-l+1,0]} - u(l)) I(n \geq N_l) &= \mathbf{E}(\theta^l Y_{[-l+1,0]} - u(l)) \\ &\quad - \mathbf{E}\{(\theta^l Y_{[-l+1,0]} - u(l)) I(n < N_l)\}. \end{aligned}$$

But

$$\mathbf{E}(\theta^l Y_{[-l+1,0]} - u(l)) = \mathbf{E}(Y_{[-l+1,0]}) - l\lambda^{-1}$$

and since $Z_{[-n,0]} \to \infty$ a.s.,

$$\lim_{n\to\infty} \mathbf{E}\{(\theta^l Y_{[-l+1,0]} - u(l)) I(n < N_l)\} = 0$$

and

$$\lim_{n\to\infty} \mathbf{E}\{S_{[1,l]} I(n < N_l)\} = 0.$$

Since

$$0 \leq \mathbf{E}\{(\theta^l Y_{[-l+1,0]} - u(l)) I(n \geq N_l)\} + \mathbf{E}\{S_{[1,l]} I(n < N_l)\}$$

for all n, l and since the right-hand side converges to $\mathbf{E}Y_{[-l+1,0]} - l\lambda^{-1}$, we get

$$\mathbf{E}Y_{[-l+1,0]} - l\lambda^{-1} \geq 0 \tag{84}$$

Therefore $\lambda^{-1} \leq \mathbf{E}Y_{[-l+1,0]}/l$ for all $l \geq 1$ and $\lambda^{-1} \leq \lim_{l\to\infty} \mathbf{E}Y_{[-l+1,0]}/l \equiv \gamma(0)$. ■

Theorem 14 *If $\rho > 1$, then $\gamma(C) > 0$ for all $1 \geq C > 0$. In particular, if $\rho > 1$ then $Z_{[-n,0]} \equiv Z_{[-n,0]}(1) \to \infty$ a.s. as n tends to ∞ (in other words $\mathbf{P}(A) = 1$).*

Proof The monotonicity properties imply

$$Z_{[-n,0]}(C) \geq Z_{[-n,0]} \geq Y_{[-n,0]} + u(-n)$$

a.s. for all $n \geq 0$ and $C \in [0,1]$. So

$$\lim_{n\to\infty} \frac{Z_{[-n,0]}(C)}{n} \geq \gamma(0) - \lambda^{-1} > 0, \quad a.s.$$

■

For $C \geq 0$, let

$$\delta(C) = \lim_{n\to\infty} X_n(C)n. \tag{85}$$

¿From the definition,

$$\delta(C) = \gamma(C) + C\lambda^{-1}. \tag{86}$$

The monotonicity properties and Lemma 4 imply that $\delta(C)$ is a continuous and non-decreasing function of C.

Remark 22 *The results of Theorems 13-14 and of Theorems 15-17 below are still true for the networks with multi-server stations (see Remark 20).*

5.2 Computation of $\gamma(0)$

Let

$$b = \max_{1\leq k\leq K} b^k, \tag{87}$$

where

$$b^k = \mathbf{E}[S^k(0)]. \tag{88}$$

Lemma 5 *In the case $K = 1$, for all $C \geq 0$*

$$\delta(C) = \max(b, C\lambda^{-1}) \equiv \max(b^1, C\lambda^{-1}). \tag{89}$$

Proof For $K = 1$, the network boils down to a single-server queue with feedback. Since the service times and the switching decisions are associated with stations, the workload in this model is equivalent to that of a single-server queue without feedback and with service times $S^1(n), n \geq 1$. ■

Lemma 6 *For all $C \geq 0$, for each fixed $K \geq 1$*

$$\delta(C) \geq \max(b, Ca). \tag{90}$$

Proof Let k be such that $b^k = b$. Consider an auxiliary sequence of networks $\tilde{\Sigma}_{[m,n]}$ with the same interarrival times and the same switching decisions, and with service times

$$\tilde{\sigma}_j^k = \sigma_j^k \tag{91}$$

for $-\infty < j < \infty$ and

$$\tilde{\sigma}_j^i = 0 \tag{92}$$

for $i = 1, \ldots, K,\ i \neq k$ and $-\infty < j < \infty$.

This boils down to a sequence of single-server queues, and the equality

$$\tilde{\delta}(C) = \max(b, Ca) \tag{93}$$

follows from Lemma 5. The monotonicity properties imply

$$\delta(C) \geq \tilde{\delta}(C) = \max(b, Ca). \tag{94}$$

■

Corollary 12 *$\gamma(0) \equiv \delta(0) \geq b$.*

Theorem 15 *For all Euler network*

$$\gamma(0) = \delta(0) = b. \tag{95}$$

Proof The proof is given in § 5.3. As it happens, it is simpler to prove a more general result, and we shall in fact prove Theorem 15 for the networks with *bulk arrivals*. ■

Corollary 13 *If $\rho \equiv b\lambda^{-1} < 1$ then for each $k \in \{1, \ldots, K\}; l \in \{1, \ldots, K, K+1\}$, the process $\{\Gamma^{k,l}_{[-n,0]}(t), t \geq 0\}$ converges (monotonically increasing) to some finite limit $\{\Gamma^{k,l}(t), t \geq 0\}$ a.s.*

5.3 Networks with Bulk Arrivals

For each $-\infty < n < \infty$, let

$$\Sigma(n) \equiv \{N(n), T(n), \sigma(n), \nu(n)\} \tag{96}$$

be the composition of $N(n)$ simple networks, where $N(n)$ is a random variable and where $T(n) = (t(n, 1), \ldots, t(n, N(n)))$. We will denote $d^{k,l}(n)$ the total number of events from station k to l in this Euler network. For all $m \leq n$, the network $\Sigma_{(m,n)}$ is defined as the composition $\Sigma(m) + \ldots + \Sigma(n)$. Our notations will parallel those of $\Sigma_{[m,n]}$ for the composition of simple networks (e.g. $Z_{(m,n)}$ will be the time to empty the system, measured from time $t(n, 1)$ etc.).

Such a sequence of networks will be said to satisfy *bulk arrival* assumptions if the sequence

$$\Xi(n) \equiv \{N(n), \tau(n), \mu(n), \sigma(n), \nu(n)\}, \tag{97}$$

(where $\tau(n) = t(n+1,1) - t(n,1)$ and $\mu(n) = (t(n,2) - t(n,1), \ldots, t(n,N(n)) - t(n,1))$) is stationary and ergodic. Note that bulk arrival statistical assumptions include the statistical assumptions of the previous sections as a special case (by taking $N(n) = 1$ for all n). Assume that

$$\tau(n), \quad d^k(n), \quad S^k(n) \equiv \sum_{j=1}^{d^k(n)} \sigma_j^k(n), \tag{98}$$

have finite first moments. Let $b^k = \mathbf{E}S^k(1)$ and $b = \max_k b^k$. Using the monotonicity and ergodicity properties of the previous sections, we obtain as above that there exist constants γ and $\gamma(C)$ such that

$$\gamma = \lim_n \frac{Z_{(-n,0)}}{n}, \quad \gamma(C) = \lim_n \frac{Z_{(-n,0)}(C)}{n}, \quad \gamma(0) = \lim_n \frac{Y_{(-n,0)}}{n}, \quad a.s.$$

Arguments similar to those of the preceding sections also imply that whenever $\lambda\gamma(0) = E[\tau(n)]^{-1}\gamma(0) < 1$, then the increasing sequence $Z_{(-n,0)}$ a.s. converges to a finite limit as n goes to ∞.

Theorem 16 *The statement of Theorem 15 holds true for networks with bulk arrivals.*

For the proof, we will assume that $\mathbf{E}d^{0,K}(1) > 0$ (if this is not true, the stations should be renumbered) and that $b^k > 0$ for all $k = 1, \ldots, K$ (otherwise we have a model which is equivalent to a network with less than K stations).

Consider the network $\Sigma_{(1,n)}(0)$. For an arbitrary fixed $d > 0$, introduce new sequences of service times:

$$\hat{\sigma}_j^K(l) = \frac{(b+d)\sigma_j^K(l)}{b^K}, \quad \hat{\sigma}_j^k(l) = \frac{b\sigma_j^k(l)}{b^k}, \quad k = 1, \ldots, K-1, \; l = 1, 2, \ldots, j = 1, 2, \ldots, d^k(l).$$

Let $\hat{\Sigma}_{(1,n)}(0)$ be the same network as $\Sigma_{(1,n)}(0)$ but with service times $\{\hat{\sigma}_j^k(i)\}$ rather than $\{\sigma_j^k(i)\}$. The monotonicity property implies that $\gamma(0) \leq \hat{\gamma}(0)$. So if we prove the inequality $\hat{\gamma}(0) \leq b + d$, this together with Corollary 12 will complete the proof of the statement of Theorem 16, since $d > 0$ is arbitrary. Till the end of the present subsection, we will work on the networks $\hat{\Sigma}_{(1,n)}(0)$ rather than $\Sigma_{(1,n)}(0)$. For sake of notational simplicity, we will drop the '^' in what follows, which is tantamount to saying that our reference networks $\Sigma(n)$ are such that $b^K = b + d$ and $b^k = b$ for $k = 1, \ldots, K-1$. For n fixed, we will use the following notations for the network $\Sigma_{(1,n)}(0) \equiv (N, T, \sigma, \nu)$, where $N \equiv N_n = N(1) + \cdots + N(n)$, σ^k is the concatenation of $\sigma^k(1), \ldots, \sigma^k(n)$ and ν^k is the concatenation of $\nu^k(1), \ldots, \nu^k(n)$.
We will denote $\Psi_i^{k,l}$ the first order variables of this network; we will also use the following notations:

$$F^{k,l}(n) = \sum_{i=1}^{n} d^{k,l}(i), \quad F^k(n) = \sum_{l=1}^{K+1} F^{k,l}(n), \tag{99}$$

for $n \geq 1$, with $F^{k,l}(0) = 0$, and

$$v(n) = \sum_{i=1}^{n} S^K(i), \tag{100}$$

for $n \geq 1$, with $v(0) = 0$.

The proof is based on the construction of an auxiliary network and two delayed networks, all associated with $\{\Sigma(n)\}$. The auxiliary network is obtained by considering nodes $\{1, \ldots, K-1\}$ as a first subnetwork and node K as a second one, and by replacing the transitions from one subnetwork to the other by external arrivals. The epochs at which the arrivals take place are described in the following formal definition:

Auxiliary network

Let $\tilde{\Sigma}(n) = (\tilde{N}(n), \tilde{T}(n), \tilde{\sigma}(n), \tilde{\nu}(n))$ be the Euler network with the following characteristics:

- $\tilde{N}(n) = \sum_{k=1}^{K} d^{0,k}(n) + \sum_{k=1}^{K-1} d^{k,K}(n) + \sum_{k=1}^{K+1} d^{K,k}(n)$ ($\tilde{N}(n)$ is the number of original external arrivals, plus that of all internal transitions to station K, plus the number of departures from station K). Note that $\tilde{N}(n) = d^K(n) + p(n)$, where $p(n) \equiv \sum_{k=1}^{K-1}(d^{0,k}(n) + d^{K,k}(n)) + d^{K,K+1}(n) = \sum_{k=1}^{K} d^{0,k}(n) + \sum_{k=1}^{K-1} d^{k,K}(n)$;
- $\tilde{T}(n) = (v(n-1), \ldots, v(n-1), s(1,n), \ldots, s(d^K(n), n))$, with $v(n-1)$ occurring $p(n)$ times and with $s(j,n) = v(n-1) + \sum_{i=1}^{j} \sigma_i^K(n)$, $1 \leq j \leq d^K(n)$, so that $s(d^K(n), n) = v(n)$;
- $\tilde{\sigma}(n) = \sigma(n)$;
-
  - $\tilde{\nu}_j^K(n) = K + 1$, for all $j = 1, \ldots, d^K(n)$;
  - $\tilde{\nu}_j^k(n) = \nu_j^k(n)$ if $\nu_j^k(n) \neq K$ and $\tilde{\nu}_j^k(n) = K + 1$ if $\nu_j^k(n) = K$, for $k = 1, \ldots, K-1$, $j = 1, \ldots, d^k(n)$.
  -
    - $\tilde{\nu}_j^0(n) = K$ for $1 \leq j \leq d^{0,K}(n) + \sum_{k=1}^{K-1} d^{k,K}(n)$;
    - for $1 \leq k \leq K-1$, $\tilde{\nu}_j^0(n) = k$, for all $d^{0,K}(n) + \sum_{k=1}^{K-1} d^{k,K}(n) + \sum_{l=1}^{k-1} d^{0,l}(n) < j \leq d^{0,K}(n) + \sum_{k=1}^{K-1} d^{K,k}(n) + \sum_{l=1}^{k} d^{0,l}(n)$;
    - $\tilde{\nu}_{p(n)+j}^0(n) = \nu_j^K(n)$, for $1 \leq j \leq d^K(n)$.

So in $\tilde{\Sigma}(n)$, a bulk of size $p(n)$ arrives at time $v(n-1)$, which brings $d^{0,K}(n) + \sum_{k=1}^{K-1} d^{k,K}(n)$ customers to station K, and $d^{0,k}(n)$ customers to station k, $k = 1, \ldots, K-1$; the remaining $d^K(n)$ external arrivals form a point process on the time interval $(v(n-1), v(n)]$, which is the same (both in terms of epochs and switching) as the output point process from station K when saturated.

The auxiliary network, $\tilde{\Sigma}_{(1,n)} \equiv (\tilde{N}, \tilde{T}, \tilde{\sigma}, \tilde{\nu})$, is defined as the composition $\tilde{\Sigma}_{(1,n)} = \tilde{\Sigma}(1) + \ldots + \tilde{\Sigma}(n)$. ■

Remark 23 *Using the assumption that $\Sigma(n)$ is Euler and considering the $N(n)$ routes which generate its switching sequences, it is easy to check that $\tilde{\Sigma}(n)$ is indeed an Euler network with $\tilde{N}(n)$ routes and that*

$$\tilde{d}^k(n) = d^k(n), \quad \forall k. \tag{101}$$

So $\tilde{\Sigma}_{(1,n)}$ is Euler.

Remark 24 *In $\tilde{\Sigma}_{(1,n)}$, the routes originating from station K, which only involve node K, are completely disjoint from those originating from $k \in \{1, \ldots, K-1\}$, which never involve node K. So the network $\tilde{\Sigma}_{(1,n)}$ can be seen as the juxtaposition of two* disconnected *Euler bulk arrival (i.e. each of them satisfies the stationarity and ergodicity assumptions of (97)-(98)): subnetworks $\{\tilde{\Sigma}'_{(1,n)}\}$ and $\{\tilde{\Sigma}''_{(1,n)}\}$*

- *Subnetwork $\tilde{\Sigma}'_{(1,n)}$, with $K-1$ stations, with bulk interarrival times $\tilde{\tau}'(i) = S^K(i)$ (with mean value $b^K = b + d$), with $\tilde{\mu}'(i) = (0, \ldots, 0, s(1,n) - v(n-1), \ldots, s(d^K(n), n) - v(n-1))$, and with service parameters*

$$b'^k = \mathbf{E}\Big(\sum_{j=1}^{\tilde{d}^k(1)} \sigma_j^k\Big) = \mathbf{E}\Big(\sum_{j=1}^{d^k(1)} \sigma_j^k\Big) = b^k = b, \quad k = 1, \ldots, K-1,$$

where we used (101) to obtain the second equality.

- *Subnetwork $\tilde{\Sigma}''_{(1,n)}$ with one station K (i.e. a single-server queue with bulk arrivals). Note that this station is never empty in this network (indeed, when the $d^{0,K}(n) + \sum_{k=1}^{K-1} d^{k,K}(n)$ customers which arrive at time $v(n-1)$ have completed their services, there are $d^{K,K}(n)$ more customers waiting in queue K; in addition, $d^{0,K}(n) + \sum_{k=1}^{K} d^{k,K}(n) = d^K(n)$).*

Before defining the delayed networks, we return to the reference network $\Sigma_{(1,n)}(0)$ for a few more definitions. Let

$$a_n = F^{0,K}(n) = \sum_{i=1}^{n} d^{0,K}(i) \tag{102}$$

and

$$m_n = \max\{m \geq 1 : F^K(m) \leq a_n/2\}, \tag{103}$$

where $m_n = 0$ if $d^K(1) > a_n/2$. Note that during the time interval $(0, v(m_n))$, station K is never empty in the network $\Sigma_{(1,n)}(0)$.

Remark 25 *In $\Sigma_{(1,n)}(0)$, all external arrivals take place at time 0. Thus, in view of Remark 13, replacing ν^0 by an arbitrary permutation of this sequence leads to the same Γ process.*

First delayed network

Let $\dot{\Sigma}_{(1,n)}$ be the network which only differs from $\Sigma_{(1,n)}(0)$, through the T and ν^0 variables:

- $T_n = (0, \ldots, 0)$ is replaced by $\dot{T}_n = (0, \ldots, 0, v(0), \ldots, v(0), v(1), \ldots, v(1), \ldots, v(m_{n-1}), \ldots, v(m_{n-1}), v(m$ where v was defined in (100); the multiplicity of 0 is a_n and that of $v(j)$ is $\sum_{k=1}^{K-1} d^{0,k}(j+1)$ for $j < m_n$ and $\sum_{l=m_n+1}^{n} \sum_{k=1}^{K-1} d^{0,k}(l)$ for $j = m_n$.

- ν^0 is replaced by $\dot{\nu}^0$, where $\dot{\nu}^0_j = K$ for $1 \leq j \leq a_n$ and $\{\dot{\nu}^0_j\}_{j=a_n+1}^{F^0(n)}$ is the subsequence of ν^0 obtained by 'removing' the a_n K's of this sequence.

So a_n customers arrive in station K at time 0 and for $j \leq v(m_n)$, $d^{0,k}(j)$ customers arrive in station k at time $v(j-1)$. ■

Remark 26 *The network $\dot{\Sigma}_{(1,n)}$ is also such that station K never empties in $(0, v(m_n))$. Therefore, on this time interval, for all $k = 1, \ldots, K-1$, the superposition of the (departure) point process from station K to station k and of the point process from station 0 to station k in $\dot{\Sigma}_{(1,n)}$ coincides with the point process from station 0 to station k in $\tilde{\Sigma}_{(1,n)}$. But $\dot{\sigma}^k = \tilde{\sigma}^k$ for all k and similarly, if one merges nodes K and 0, $\dot{\nu}^k = \tilde{\nu}^k$ for all $k = 1, \ldots, K-1$. Therefore, on the time interval $(0, v(m_n))$, the $\bar{\Gamma}^k(t)$ processes of $\dot{\Sigma}_{(1,n)}$ and $\tilde{\Sigma}_{(1,n)}$ coincide for all $k = 1, \ldots, K-1$.*

Remark 27 *In $\tilde{\Sigma}_{(1,n)}$, the total number of services completed on station k at time $(v(m_n)-)$ cannot exceed $\sum_{i=1}^{m_n} \tilde{d}^k(i) = \sum_{i=1}^{m_n} d^k(i) = F^k(m_n)$ (see (101)). This and the preceding remark imply that the total number of services completed in $\dot{\Sigma}_{(1,n)}$ at time $(v(m_n)-)$ cannot exceed $F^k(m_n)$. Therefore, necessarily $\dot{\Psi}^k_{F^k(m_n)+1} \geq v(m_n)$ for all $k = 1, \ldots, K$.*

Second delayed network

Our second delayed network $\ddot{\Sigma}_{(1,n)}$ is the network obtained by delaying (see Remark 11) the network $\dot{\Sigma}_{(1,n)}$ with the sequence $\{\alpha^k_j\}$, $j = 1, \ldots, F^k(n)$, which is defined as follows:

- for $k = 1, \ldots, K$
  - for $j \leq F^k(m_n)$, $\alpha^k_j = -\infty$;
  - for $j > F^k(m_n)$, $\alpha^k_j = U$, where U is some variable larger than $v(m_n)$, to be determined later.
- for $k = 0$
  - $\alpha^0_j = 0$ if $\dot{\Psi}^0_j < v(m_n)$;
  - $\alpha^0_j = U$ if $\dot{\Psi}^0_j \geq v(m_n)$;

■

Remark 28 *Consider two Euler networks Σ and Σ' having the same number of events on each station. We will say that Σ' is a majorant of Σ if the j-th event on station k takes place later in Σ' than in Σ. Network $\dot{\Sigma}_{(1,n)}$ is a majorant of $\Sigma_{(1,n)}(0)$ (this follows from the monotonicity property and Remark 25); similarly $\ddot{\Sigma}_{(1,n)}$ is a majorant of $\dot{\Sigma}_{(1,n)}$ (see Remark 16). Therefore $\ddot{\Sigma}_{(1,n)}$ is a majorant of $\Sigma_{(1,n)}(0)$.*

The variables $\ddot{\Psi}^k_{F^k(m_n)}$ associated with network $\ddot{\Sigma}_{(1,n)}$ clearly depend on U. The following property holds:

Lemma 7 *If we take $U \geq \tilde{X}_{(1,m_n)}$, (i.e. the time to empty the network $\tilde{\Sigma}_{(1,m_n)}$), then*

$$\max_{k=1,\ldots,K} \ddot{\Psi}^k_{F^k(m_n)} \leq U. \tag{104}$$

Proof It follows from Remark 27 that delaying the j-th event on station k for all $j \geq F^k(m_n + 1)$ of U such that $U \geq v(m_n)$ has no effect on the events in the time interval $(0, v(m_n))$. Therefore for any $U \geq v(m_n)$, $\ddot{\Sigma}_{(1,m_n)}$ is such that station K is never empty in $(0, v(m_n))$ and always inactive in $(v(m_n), U)$. The rest of the proof is similar to the argument used in Remark 26:

- for all $k = 1, \ldots, K - 1$, the point process from $\{0, K\}$ to k in $\ddot{\Sigma}_{(1,n)}$ and the point process from 0 to k in $\tilde{\Sigma}_{(1,m_n)}$ coincide;

- when merging 0 and K, for all $k \neq K$, the prefix $\ddot{\nu}_j^k, j = 1, \ldots, F^k(m_n)$ of the switching sequence used in $\ddot{\Sigma}_{(1,n)}$ coincides with the sequence $\tilde{\nu}_j^k, j = 1, \ldots, F^k(m_n)$ used in $\tilde{\Sigma}_{(1,m_n)}$.

Since, in addition, the service sequences coincide, for all $t \in (0, U)$, the $\bar{\Gamma}(t)$ processes of $\ddot{\Sigma}_{(1,n)}$ and $\tilde{\Sigma}_{(1,m_n)}$ coincide for all $k = 1, \ldots, K-1$. Therefore, if we take $U \geq \tilde{X}_{(1,m_n)}$, Equation (104) holds since

$$\max_{k=1,\ldots,K} \ddot{\Psi}^k_{F^k(m_n)} = \max_{k=1,\ldots,K} \tilde{\Psi}^k_{F^k(m_n)} = \tilde{X}_{(1,m_n)} \leq U. \tag{105}$$

■

Remark 29 *It is easy to check that the number of customers present in station k at time $U \geq X_{(1,m_n)}$ in $\ddot{\Sigma}_{(1,n)}$ is exactly $F^{0,k}(n) - F^{0,k}(m_n)$: indeed, the number of services completed in queue l at time $U-$ is $F^l(m_n)$ for all $l = 1, \ldots, K$; in addition, the number of arrivals in station k by time U is $F^{0,k}(n) + \sum_{l=1}^{K} F^{l,k}(m_n)$; thus the number of customers present in station k at time U is $F^{0,k}(n) + \sum_{l=1}^{K} F^{l,k}(m_n) - F^k(m_n)$, which is equal to $F^{0,k}(n) - F^{0,k}(m_n)$ since $F^k(m_n) = \sum_{l=0}^{K} F^{l,k}(m_n)$. This and the preceding lemma show that for $U \geq \tilde{X}_{(1,m_n)}$, the network $\ddot{\Sigma}_{(1,n)}$ is actually the composition of the two* separated *Euler networks:*

$$\ddot{\Sigma}_{(1,n)} = \ddot{\Sigma}' + \ddot{\Sigma}'', \tag{106}$$

where $\ddot{\Sigma}''$ is the same Euler network as $\Sigma_{(m_n+1,n)}(0)$, but with all arrivals taking place at time U rather than 0.

Key relationship between the three networks

We summarize the basic relations between these networks below:

- $\ddot{\Sigma}_{(1,n)}$ is a majorant of $\Sigma_{(1,n)}(0)$ (Remark 28), so that

$$Y_{(1,n)} \leq \ddot{X}_{(1,n)}. \tag{107}$$

- For $U = \tilde{X}_{(1,m_n)}$, $\ddot{\Sigma}_{(1,n)} = \ddot{\Sigma}' + \ddot{\Sigma}''$, where these two networks are separated; in addition, $\ddot{\Sigma}''$ and $\Sigma_{(m_n+1,n)}(0)$ have the same Y process (Remark 29). Therefore

$$\ddot{X}_{(1,n)} = \tilde{X}_{(1,m_n)} + Y_{(m_n+1,n)}. \tag{108}$$

Thus we get from (107)-(108) that

$$Y_{(1,n)} \leq \tilde{X}_{(1,m_n)} + Y_{(m_n+1,n)} \leq v(m_n) + \tilde{Z}_{(1,m_n)} + Y_{(m_n+1,n)} \quad a.s. \tag{109}$$

Proof of Theorem 16
The proof is by induction on K. For $K = 1$, the result is that of Lemma 5. Assume that the theorem holds for networks with $K-1$ stations. Thus, we can apply the induction assumption to network $\tilde{\Sigma}'_{(1,n)}$ defined in Remark 24; since $\tilde{\lambda}'^{-1} = \mathbf{E}\tilde{\tau}_1 = (b+d)^{-1}$ is such that $\tilde{\rho}' \equiv \tilde{\lambda}' \times \max_{1 \leq k \leq K-1} \tilde{b}^k = (b+d)^{-1} b < 1$, we obtain from this that $\tilde{Z}'_{(1,n)}$ is bounded from above by a finite stationary sequence

(Corollary 9). This and the fact that the second network $\tilde{\Sigma}''_{(1,n)}$ is such that $\tilde{Z}''_{(1,n)} = S^K(n)$ allow us to state that there exists an a.s. finite stationary sequence $\{\tilde{Z}^{(n)},\ n \geq 1\}$ such that

$$\tilde{Z}_{(1,n)} \leq \tilde{Z}^{(n)}, \quad n = 1, 2, \ldots \tag{110}$$

The CLLN implies the relations:

$$\frac{a_n}{n} \to \mathbf{E}(d^{0,K}(1)) > 0, \quad \text{and} \quad m_n \to \infty, \quad a.s.$$

as $n \to \infty$. Similarly,

$$\frac{m}{F^K(m)} \to \frac{1}{\mathbf{E}d^K(1)} < \infty, \quad \frac{m}{F^k(m+1)} \to \frac{1}{\mathbf{E}d^K(1)}, \quad a.s.$$

as $m \to \infty$. Since

$$\frac{m_n}{F^K(m_n)} \geq \frac{m_n}{a_n/2} \geq \frac{m_n}{F^K(m_n+1)},$$

then

$$\frac{m_n}{n} = \frac{m_n}{a_n/2}\frac{a_n/2}{n} \to \frac{\mathbf{E}d^{0,K}(1)}{2\mathbf{E}d^K(1)} \equiv c \tag{111}$$

a.s., where $0 < c \leq 1/2$.

We have

$$\frac{v(m_n)}{n} \to (b+d)c \quad a.s.$$

In addition, we get from the relation $Y_{(m_n+1,n)}/n = (Y_{(m_n+1,n)}/(n-m_n)) \times ((n-m_n)/n)$ that

$$Y_{(m_n+1,n)}/n \to (1-c)\gamma(0)$$

in probability (see Appendix 8.3). Finally,

$$\frac{\tilde{Z}_{(1,m_n)}}{n} \to 0$$

in probability as $n \to \infty$ (see Appendix 8.3). Therefore (109) implies the inequality:

$$\gamma(0) \equiv \lim_n \frac{Y_{(1,n)}}{n} \leq (b+d)c + (1-c)\gamma(0)$$

for some $0 < c \leq 1/2$. Therefore

$$\gamma(0) \leq b + d. \tag{112}$$

■

6 Second Order Ergodic Results

6.1 Stochastic Recurrences

Now we are ready to write down a recurrence for constructing the (second-order) state of the Euler network $\Sigma_{[-n,m+1]}$ from that of $\Sigma_{[-n,m]}$ for each fixed n. More concretely, we want to get a representation of the form

$$W_{[-n,m+1]} = f(W_{[-n,m]}, \eta(m+1)), \tag{113}$$

where the function f is fixed (i.e. non-random and independent of n and m), $\{\eta(m)\}$ is some stationary ergodic sequence and $W_{[-n,m]}$ is the 'state' of network $\Sigma_{[-n,m]}$. Such a representation is often referred to as a stochastic recursive sequence (see Borovkov and Foss [12] or Baccelli and Brémaud [3]). There are several such representations and we will focus on one of them only.

Consider the space

$$D_+^0(K) \equiv D_+^0 \times D_+^0 \times \cdots \times D_+^0, \quad (K(K+1) \text{ times}), \tag{114}$$

endowed with the coordinate partial order $\leq$ (see § 3.7 for the partial order on D_+^0), and let

$$W_{[-n,m]} = \{(\Gamma^{1,1}_{[-n,m]}(t), \ldots, \Gamma^{1,K+1}_{[-n,m]}(t), \Gamma^{2,1}_{[-n,m]}(t), \ldots, \Gamma^{K,K+1}_{[-n,m]}(t)), t \geq 0\} \in D_+^0(K) \tag{115}$$

and

$$\eta(n) = \xi(n), \tag{116}$$

where $\xi(n)$ is defined in (58). The data of $W_{[-n,m]}$ completely defines the (residual) service times and the switching decisions of $\Sigma_{[-n,m]}$ after time $t(m)$, as shown by the following construction.

6.2 Construction of a Network from W

Let $W \equiv \{\Gamma^{1,1}(t), \ldots, \Gamma^{K,K+1}(t), \quad t \geq 0\} \in D_+^0(K)$. We will use the following notations:

$$\Gamma^k(t) = \sum_{i=1}^{K+1} \Gamma^{k,i}(t), \quad Q^k(t) = \Gamma^k(t) - \sum_{l=1}^{K} \Gamma^{l,k}(t), \quad k = 1, \ldots, K, \ t \geq 0$$

(already introduced in § 3.8), and

$$\Gamma^{k,l} \equiv \Gamma^{k,l}(0), \quad \Gamma^k \equiv \Gamma^k(0), \quad Q^k \equiv Q^k(0), \quad k, l = 1, \ldots, K+1. \tag{117}$$

Associated with W, we also define

$$P_j^{k,l} = \inf\{t \geq 0 : \quad \Gamma^{k,l}(t) < j\}, \quad 1 \leq k \leq K, \ 1 \leq l \leq K+1, \tag{118}$$

and

$$P^{k,l} \equiv \{P_j^{k,l}, \quad 1 \leq j \leq \Gamma^{k,l}\}. \tag{119}$$

The main difficulty for reconstructing the sequence of service times and switching decisions from W comes from the possibility of simultaneous departures from a station (in case of zero-valued service times on this station). For $t > 0$, let

$$H(t) = \{(k, l, j) : \quad P_j^{k,l} = t\} \tag{120}$$

and

$$J(t) = \#(H(t)). \tag{121}$$

For $t > 0$ such that $J(t) > 1$, we take an arbitrary numbering $\{(k_r, l_r, j_r), \ 1 \leq r \leq J(t)\}$ of the elements of $H(t)$ satisfying the following constraints: if $k_{r_1} = k_{r_2}$ and $r_1 < r_2$, then $j_{r_1} < j_{r_2}$. For such a numbering, let $\{Q_r^k(t), \ k = 1, \ldots, K, \ r = 0, \ldots, J(t)\}$ be the sequence defined by:

$$\begin{aligned} Q_0^k(t) &= Q^k(t-), \\ Q_r^k(t) &= Q_{r-1}^k(t) + I_{\{l_r = k\}} - I_{\{k_r = k\}}, \quad \text{for } r = 1, \ldots, J(t). \end{aligned} \tag{122}$$

Note that $Q^k_{J(t)}(t) = Q^k(t)$ for all k.

Definition 8 *We say that $W \in D_+^0(K)$ is* admissible*, if for each $t \geq 0$, $k = 1, \dots, K$,*

(i) $Q^k(t) \geq 0$,

(ii) for each $t > 0$ with $J(t) > 1$, there exists a numbering of the elements of $H(t)$ satisfying the above constraints and such that $Q^k_{r-1}(t) \geq I_{\{k_r = k\}}$ for all $k = 1, \dots, K$, $r = 1, \dots, J(t)$.

We will denote $\hat{D} \subseteq D_+^0(K)$ the admissible subspace of $D_+^0(K)$.

If the function W is that associated with an Euler network Σ as in (115), then W is in $\hat{D}$; so, in what follows, we will restrict ourselves to the case when W is admissible. For $1 \leq k \leq K$ and $t > 0$, let

$$\begin{aligned} j_k(t) &= \Gamma^k - \Gamma^k(t-), \\ a(t) &= \sup\{0 \leq u < t : \quad \Gamma^k(u) < \Gamma^k(u-)\}, \\ b(t) &= \sup\{0 \leq u < t : \quad Q^k(u) = 0\} \end{aligned}$$

(here $a(t) = 0$ and $b(t) = 0$ if these sets are empty). With W as above, we associate a network $\Sigma(W) = (N, T, \sigma, \nu)$ with finite number of customers $N = \sum_k Q^k$, with zero-valued arrival epochs $T = (0, \dots, 0)$; its sequences σ and ν, with $\nu^0 = \{\nu_i^0\}_{i=1}^N$, $\nu^k = \{\nu_i^k\}_{i=1}^{\Gamma^k}$ and $\sigma^k = \{\sigma_i^k\}_{i=1}^{\Gamma^k}$, $k = 1, \dots, K$, are defined as follows:

- $\nu_i^0 = k$ for $\sum_{l=1}^{k-1} Q^l < i \leq \sum_{l=1}^{k} Q^l$, $k = 1, \dots, K$;
- If for some $t > 0$ and $k = 1, \dots, K$, there exists only one $l = 1, \dots, K+1$ and one $i \geq 1$ such that $P_i^{k,l} = t$ (i.e. $\Gamma^{k,l}(t) = \Gamma^{k,l}(t-) - 1$), then $\nu^k_{j_k(t)+1} = l$ and $\sigma^k_{j_k(t)+1} = t - \max(a(t), b(t))$;
- If for some $t > 0$ and $k = 1, \dots, K$, there exist $q \geq 2$ numbers $l_1, \dots, l_q$ such that $P_{i_1}^{k,l_1} = t, \dots, P_{i_q}^{k,l_q} = t$, for some $i_1, \dots, i_q$ with $i_p = i_1 + p - 1$ (i.e. if $\Gamma^k(t) = \Gamma^k(t-) - q$). Since W is admissible, we can assume that the chosen numbering is such that (ii) is satisfied. We then take $\sigma^k_{j_k(t)+1} = t - \max(a(t), b(t))$, $\sigma^k_{j_k(t)+d} = 0$ for $d = 2, \dots, q$ and $\nu^k_{j_k(t)+1} = l_1, \dots, \nu^k_{j_k(t)+q} = l_q$.

By this construction, we associate a family of networks (depending on the choice of admissible numbering that is made) with W; all these networks are equivalent in that they share the same Γ-processes.

Definition 9 *We will denote $\prec$ the partial order defined on the space $\hat{D}$ by $W \prec \tilde{W}$ if there exists a pair of networks $\Sigma \equiv \Sigma(W)$ and $\tilde{\Sigma} \equiv \Sigma(\tilde{W})$, associated with W and $\tilde{W}$, respectively, and such that*

- *$\Gamma^{k,l} \leq \tilde{\Gamma}^{k,l}$ for all k, l;*
- *$\nu^k_{\Gamma^k - j} = \tilde{\nu}^k_{\tilde{\Gamma}^k - j}$ for all $k = 1, \dots. K$, $0 \leq j < \Gamma^k$;*
- *$\sigma^k_{\Gamma^k - j} \leq \tilde{\sigma}^k_{\tilde{\Gamma}^k - j}$ for all $k = 1, \dots, K$, $0 \leq j < \Gamma^k$.*

Lemma 8 *If $W \prec \tilde{W}$ in $\hat{D}$, then $W \leq \tilde{W}$, i.e. $\Gamma^{k,l}(t) \leq \tilde{\Gamma}^{k,l}(t)$ for all k, l, t.*

Proof Let Σ' be the network with the same data as $\Sigma(\tilde{W})$, but for the service times, which are replaced by ${\sigma'}_j^k = 0$ for $j = 1, \ldots, \tilde{\Gamma}^k - \Gamma^k$-1, and ${\sigma'}_j^k = \sigma_j^k$ for $j = \tilde{\Gamma}^k - \Gamma^k, \ldots, \tilde{\Gamma}^k$. Using the monotonicity property of Lemma 2, we obtain that ${\Gamma'}^{l,k}(t) \leq \tilde{\Gamma}^{l,k}(t)$ for all l, k, t. The proof is concluded when observing that ${\Gamma'}^{l,k}(t) = \Gamma^{l,k}(t)$ for all $t > 0$, and that $\Gamma^{k,l}(0) \leq \tilde{\Gamma}^{k,l}(0)$ by assumption. ■

Lemma 9 *Let $\{W^{(m)}\}$ be a $\prec$-increasing sequence of $\hat{D}$ such that the pointwise limit $W \equiv \lim_m W^{(m)}$ belongs to $D^0_+(K)$. Then W belongs to $\hat{D}$ and $W^{(m)} \prec W$ for all m.*

Proof For each t, $\Gamma^{k,l}(t) \equiv \lim_m \Gamma^{(m),k,l}(t) < \infty$, and all these functions are piecewise constant and integer-valued. Therefore, there exists a finite number $L(t)$ such that $\Gamma^{k,l}(t) = \Gamma^{(m),k,l}(t)$ for all $m \geq L(t)$ and for each k, l. Therefore $Q^k(t) \geq 0$ for each k and t. The property (ii) follows from the fact that we can choose a family of networks $\Sigma(W^{(m)})$ and a network $\Sigma(W)$ such that $\nu_j^{(m),k,l} = \nu_j^{k,l}$ for all $m \geq L(0)$ and for each k, l and $j = 1, \ldots, \Gamma^{k,l}$. ■

6.3 Main Second-Order Ergodic Theorems

So, for each $W \in \hat{D}$ and for each $\xi \in \Xi$, where Ξ is the set of vectors of the form (58), we can define $f(W, \xi)$ as the vector of Γ-processes of the composition of the networks $\Sigma(W)$ and Σ', where Σ' is the simple network associated with ξ. It follows from the monotonicity property that the function $f : \hat{D} \times \Xi \to \hat{D}$, defined in (113), is monotone in its first argument, with respect to $\prec$: if $W \prec \tilde{W}$, then $f(W, \xi) \prec f(\tilde{W}, \xi)$. The monotonicity properties, Theorem 13 and Corollary 10 imply the following result:

Theorem 17 *If $\rho < 1$, then for each integer $-\infty < m < \infty$, the sequence $W_{[-n,m]}$ converges $\prec$-monotonically and almost surely (as $n \to \infty$) to a finite random variable $W(m) \in \hat{D}$, and $\{W(m)\}$ is a stationary ergodic sequence such that*

$$W(m) = W(0) \circ \theta^m \tag{123}$$

and

$$W(m+1) = f(W(m), \xi(m+1)), \tag{124}$$

for all m. Moreover $\{W(m)\}$ is the minimal stationary solution of the equation (124): if $\tilde{W}(m+1) = f(\tilde{W}(m), \xi(m+1))$ is another stationary solution, then $W(m) \prec \tilde{W}(m)\ \forall m$ a.s. so that in particular $W(m) \leq \tilde{W}(m)\ \forall m$ a.s. (where $\leq$ is the coordinate partial order on $D^0_+(K)$).

Proof The main thing to prove is (124), as the last assertion follows by monotonicity. Observe first that

$$W_{[-n,m+1]} \prec f(\lim_l W_{[-l,m]}, \xi(m+1)) \tag{125}$$

for all n. Therefore

$$W(m+1) \equiv \lim_n W_{[-n,m+1]} \prec f(W(m), \xi(m+1)) \tag{126}$$

Since $\Gamma^{k,l}_{[-n,m]} \equiv \Gamma^{k,l}_{[-n,m]}(0)$ are integer-valued, there exists an a.s. finite random number L such that

$$\Gamma^{k,l}_{[-n,m]} = \Gamma^{k,l}_{[-n-1,m]} \tag{127}$$

for all $n > L$ and for each k, l. Therefore the inequality

$$W(m+1) \prec f(W(m), \xi(m+1)) \tag{128}$$

follows immediately from the continuity property (see Corollary 4). ■

Let $\Gamma^{k,l}(m,t),\ t \geq 0$ be the coordinates of the random variable $W(m)$ and, for each m, k, let $\{Q^k(m,t), t \geq 0\}$ be the associated residual queueing process, $Q(m,t)$ be the process $(Q^1(m,t), \ldots, Q^K(m,t))$ and $\{\chi^k(m,t),\ t \geq 0,\ k = 1, \ldots, K\}$ be the residual service-time process.

Corollary 14 *If $\rho < 1$, then for each m, k, the processes $\{Q^k_{[-n,m]}(t), t \geq 0\}$ converge a.s. (w.r. to the metric d) to the process $\{Q^k(m,t), t \geq 0\}$ as $n \to \infty$.*

Corollary 15 *If $\rho < 1$ then for each $-\infty < m < \infty,\ t \geq 0$, the vectors*

$$(Q^1_{[-n,m]}(t), \ldots Q^K_{[-n,m]}(t),\ \chi^1_{[-n,m]}(t), \ldots, \chi^K_{[-n,m]}(t))$$

converge weakly to the vector

$$(Q^1(m,t), \ldots, Q^K(m,t),\ \chi^1(m,t), \ldots, \chi^K(m,t))$$

as $n \to \infty$.

Consider the network $\Sigma = \Sigma(1) + \Sigma(2) + \ldots$, with infinite input sequence $t(1), t(2), \ldots$ Let

$$\bar{Q}(t) \equiv \{\bar{Q}^1(t), \ldots, \bar{Q}^K(t),\ t \geq 0\}$$

be the queue-length process for this network, and for each $n \geq 1$, let

$$\{Q_n(t) \equiv Q(t + t(n)),\ t \geq 0\}$$

be the residual queue-length process. Define now the process

$$Q^{(0)}(t) = \begin{cases} Q(0,t) & \text{for } 0 \leq t < t(1), \\ Q(l, t - t(l)) & \text{for } t(l) \leq t < t(l+1),\ l = 1, 2, \ldots \end{cases} \tag{129}$$

We also define the processes

$$\{Q^{(n)}(t) = Q^{(0)}(t + t(n)), t \geq 0\}, \quad n = 1, 2, \ldots. \tag{130}$$

It follows from Corollary 19 that the sequence $\{Q^{(n)}(t), t \geq 0\}_{n=0}^{\infty}$ is stationary and ergodic (in n).

Corollary 16 *If $\rho < 1$, then*

$$0 \leq Q_n(t) \leq Q^{(n)}(t) \tag{131}$$

a.s. for all $n \geq 0,\ t \geq 0$ and the processes $\{Q_n(t) \circ \theta^{-n},\ t \geq 0\}$ converge monotonically a.s. to the process $Q^{(0)}(t)$.

Proof The proof is similar to that of Corollary 9. ■

Note that the ergodicity stationarity properties imply the existence of the following limits:

$$\begin{aligned}\lim_t \frac{1}{t}\int_{x=0}^t I\{Q^{(0)}(x)\in .\}dx &= \lim_t \frac{1}{t}\int_{x=0}^t \mathbf{P}\{Q^{(0)}(x)\in .\}dx \\ &= \frac{\mathbf{E}\{\int_{x=0}^{u(1)} I\{Q^{(0)}(x)\in .\}dx\}}{\mathbf{E}u(1)} \quad a.s. \end{aligned} \tag{132}$$

(where $0/0$ means 0, by convention).

Corollary 17 *If $\rho<1$, then*

$$\lim \frac{1}{t}\int_{x=0}^t I\{\theta^{-n}\circ Q_n(x)\in .\}dx = \frac{\mathbf{E}\{\int_{x=0}^{u(1)} I\{Q^{(0)}(x)\in .\}\}}{\mathbf{E}u(1)} \quad a.s. \tag{133}$$

as $n\to\infty$.

For each scaling factor $C>0$, consider the network $\Sigma_{[-n,m]}(C)$. It follows from Theorem 17 that for each $C>b\lambda$, there exists a stationary ergodic sequence $W(m,C)$ such that

$$W(m+1,C)=f(W(m,C),\xi(m+1,C)). \tag{134}$$

In addition, $\{W(m,C)\}$ is the minimal solution of (134).

Theorem 18 *If $\rho<1$, then for each m*

$$W(m,C)\nearrow W(m)\equiv W(m,1) \tag{135}$$

a.s. as $C\searrow 1$.

Proof The proof is similar to that of Theorem 17. ■

6.4 General Initial Conditions

The notations are those of § 2.1; we will assume that $t(0)=0$. Consider an arbitrary network V with a.s. finite input sequence and finite number of services on each station, and such that all arrival epochs are non-positive. For any integer $n\geq 1$, and for any positive real number C, let ${}_V\Sigma_{[1,n]}(C)$ be the network

$${}_V\Sigma_{[1,n]}(C)=V+\Sigma_{[1,n]}(C).$$

We shall say that V is an *initial condition* for $\Sigma_{[1,n]}(C)$ and call the customers of network V *initial* customers. We shall use the following notations:

- ${}_V\Sigma_{[1,n]}\equiv {}_V\Sigma_{[1,n]}(1)$;
- ${}_V\Gamma^{k,l}_{[1,n]}(C)(t)$, ${}_VW_{[1,n]}(C)$ etc. for the characteristics of ${}_V\Sigma_{[1,n]}(C)$;

- ${}_V\Gamma^{k,l}_{[1,n]}(t),\ {}_VW_{[1,n]}$ etc. for the characteristics of ${}_V\Sigma_{[1,n]}$.

Lemma 10 *If $\rho < 1$ then for all random initial conditions V and for all real number C such that $b\lambda < C < 1$, one can define an a.s. finite random variable $\beta \equiv \lambda(C, V)$, such that, for all $n \geq 1$,*

$$
{}_VW_{[1,n]} \leq W(n, C), \quad a.s. \tag{136}
$$

on the event $\{\beta \leq n\}$, where $W(n, C)$ is the r.v. defined in (134).

Proof Let B_V be the first non-negative time when network V is empty, and let

$$
\beta = \min\{n \geq 1 : (1 - C)t_n \geq B_V\} < \infty \quad a.s. \tag{137}
$$

For each $n \geq 1$, consider the network

$$
{}_V\tilde{\Sigma}_{[1,n]}(C) = V + \tilde{\Sigma}_1 + \ldots + \tilde{\Sigma}_n,
$$

where $\tilde{\Sigma}_n$ is the same as Σ_n but with arrival epoch

$$
\tilde{t}_i = Ct_i + (1 - C)t_n, \qquad i = 1, \ldots, n. \tag{138}
$$

The monotonicity property implies that ${}_V\tilde{\Sigma}_{[1,n]}(C)$ is a majorant of ${}_V\Sigma_{[1,n]}$. Note that for $n \geq \beta$ V and $\tilde{\Sigma}_{[1,n]}$ are separated in ${}_V\tilde{\Sigma}_{[1,n]}(C)$ and, therefore,

$$
{}_VW_{[1,n]} \leq {}_V\tilde{W}_{[1,n]}(C) = W_{[1,n]}(C) \leq W(n, C) \tag{139}
$$

a.s. on the event $\beta \leq n$. ∎

Corollary 18 *If $\rho < 1$, then for each initial condition V*

(i) the sequence $\{{}_VZ_{[1,n]}\}$ is bounded in probability;

(ii) for each $\epsilon > 0$, there exists an element $f \equiv f(\epsilon) \in D^0_+$ such that

$$
\mathbf{P}({}_V\Gamma^k_{[1,n]} \leq f) \geq 1 - \epsilon \tag{140}
$$

for all $n \geq 1,\ k = 1, \ldots, K$.

Proof It is enough to prove (i) only. Property (i) follows from the inequality:

$$
\sup_n \mathbf{P}({}_VZ_{[1,n]} > x) \leq \max_{1 \leq n \leq N} \mathbf{P}({}_VZ_{[1,n]} > x) + \mathbf{P}(\zeta > N) + \mathbf{P}(Z(1, C) > x), \tag{141}
$$

for all $x \geq 0,\ N \geq 1$ and $\lambda b < C < 1$. ∎

Remark 30 **(Maximal solution)** *It is not difficult to see that if $\rho < 1$, then the sequence $\{\tilde{W}(m)\}$ defined by*

- *$\tilde{W}(m)$ is right-continuous a.s.;*
- *$d_K(\tilde{W}(m), \lim_{C \nearrow 1} W(m, C)) = 0$*

forms a maximal stationary solution of (124) (here d_K is a metric in the space $D^0_+(K)$). In particular, $\tilde{W}(m) \leq W(m, C)$ a.s. for all C such that $\lambda b < C < 1$.

7 Coupling-Convergence

Without loss of generality, we can assume that for each n and k, the random variables $\nu_j^k(n)$ and $\sigma_j^k(n)$ are defined for all $j = 1, 2, \ldots$. Let

$$\zeta(n) = \{\tau(n),\ \{\sigma_j^k(n)\}_{j=1}^{\infty},\ k = 1, \ldots, K\}. \tag{142}$$

Consider now the following set of assumptions (referred to as (I) in what follows):

1. $\{\zeta(n)\}_{n=-\infty}^{\infty}$ is a stationary and ergodic sequence;
2. the sequences $\{\{\nu_j^k(n)\}_{j\geq 1},\ k = 0, 1, \ldots, K,\ -\infty < n < \infty\}$ are mutually independent and independent of the sequence $\{\zeta(n)\}_{n=-\infty}^{\infty}$;
3. $\{\nu_j^k(n)\}_{j\geq 1}^{\infty}$ is an i.i.d. sequence, for all $k = 0, 1, \ldots, K; -\infty < n < \infty$.

Remark 31 *Let $p^{k,1}, p^{k,2}, \ldots, p^{k,K+1}$ be the law of ν_1^k, $k = 0, \ldots, K$. It is easy to check that if the routing Markov chain $P = \{p^{k,l}\}$ on $\{0, 1, \ldots, K, K+1\}$, is 'without capture' (i.e. for all k such that $p^{0,k} > 0$, there exists a sequence $0 = k_0, k_1, \ldots, k_n, k_{n+1} = K+1$ such that $p^{k_i,k_{i+1}} > 0$, for all $i = 0, \ldots, K$), then the i.i.d. sequences $\{\nu_j^k\}_{j=1}^{\infty}$ are Euler in the following sense: there exist sequences of integers $d^k(1) \leq d^k(2) \leq \ldots$ which tends to ∞ and such that if $F^k(n) \equiv \sum_{i=1}^{n} d^k(i)$, then for all n, the switching sub- sequence $\{\nu_j^k\}_{j=1}^{F^k(n)}$ is Euler. So the above assumptions fall in the framework of §4.1.*

Remark 32 *Note that under condition (I)*

$$b^k = \mathbf{E}[\sum_{i=1}^{\varphi^k(1)} \sigma_i^k(1)] = \sum_{i=1}^{\infty} \mathbf{E}[\sigma_i^k]\mathbf{P}[\varphi^k(1) \geq i]. \tag{143}$$

Theorem 19 *Assume that $\rho < 1$. Then, under condition (I),*

(i) the sequence $\{W(n)\}$ is the unique solution of (124);

(ii) one can define all the driving sequences on some probability space in such a way that the sequence $\{_V W_{[1,n]}\}$ coupling-converges to the sequence $\{W(n)\}$, for each initial condition V, i.e.

$$\mathbf{P}[_V W_{[1,l]} = W(l),\ l = n, n+1, \ldots] \to 1 \tag{144}$$

as $n \to \infty$.

Proof Note that (ii) implies (i). Indeed, if $\tilde{W}(n)$ is another stationary sequence, then we can consider the initial condition $V = \tilde{W}(0)$ (by this concise notation, we mean any network with the same Γ process as the one generating $\tilde{W}(0)$) and apply (ii). Then $\mathbf{P}[\tilde{W}(l) = W(l),\ l = n, n+1, \ldots] \to 1$ as $n \to \infty$. But both of the sequences are stationary, so $\tilde{W}(n) = W(n)$ a.s.

So the only property to prove is (ii). Let

$$p^{k,l} = \mathbf{P}(\nu_1^k(1) = l), \quad k, l = 0, \dots, K, K+1, \tag{145}$$

where $p^{0,0} = 0$, $p^{K+1,K+1} = 1$ and $p^{K+1,k} = 0$, for $k \leq K$. Consider a discrete-time Markov chain $\{R(m), m \geq 0\}$, with state space $\{0, 1, \dots, K, K+1\}$, with initial value $R(0) = 0$ and with transition matrix $\{p^{k,l}\}$. Let

$$\mu^k = \sharp\{n \geq 1 : R(n) = k\} \tag{146}$$

and

$$\pi^k = \mathbf{E}\mu^k. \tag{147}$$

Note that the random variables μ^k and $\varphi^k(1)$ have the same distribution, so $\pi^k < \infty$ for all $k = 1, \dots, K$.

In Foss [19]-[20], the following result is proved: for a given transition matrix $\{p^{k,l}\}$, one can define a matrix $\{\tilde{p}^{k,l}\}$ and a renumbering of the state space such that

1. $\tilde{p}^{k,l} = 0$, for all $0 \leq l \leq k \leq K+1$;
2. for all k, l if $\tilde{p}^{k,l} > 0$ then $p^{k,l} > 0$;
3. if $\tilde{R}(n)$ is a Markov chain with initial value $\tilde{R}(0) = 0$ and with transition probabilities $\{\tilde{p}^{k,l}\}$, then
$$\mathbf{P}(\tilde{\mu}^k \geq n) \leq \mathbf{P}(\mu^k \geq n), \tag{148}$$
for all $k = 1, \dots, K$ and $n = 1, 2, \dots$.

For all $k = 0, 1, \dots, K$ define the constant

$$C^k = \sup\{0 \leq C \leq 1 : \quad p^{k,l} \geq C\tilde{p}^{k,l} \quad \forall l = 0, \dots, K\}. \tag{149}$$

The above results imply that C^k is positive for all k. Finally, for each k, let h^k be a positive number such that $h^k < C^k$.

We now return to our network. Our aim is to construct the sequence ν on a specific probability space which is based on the above results. We first construct a sequence of mutually independent r.v.'s

$$\{\tilde{\nu}_j^k(n),\ \bar{\nu}_j^k(n),\ \beta_j^k(n), \quad k = 0, \dots, K,\ j = 1, 2, \dots,\ n = 1, 2, \dots\}$$

with the following law:

$$\begin{aligned}
\mathbf{P}(\tilde{\nu}_j^k(n) = l) &= \tilde{p}^{k,l}; \\
\mathbf{P}(\beta_j^k(n) = 1) &= 1 - \mathbf{P}(\beta_j^k(n) = 0) = h^k; \\
\mathbf{P}(\bar{\nu}_j^k(n) = l) &= \frac{p^{k,l} - h^k\tilde{p}^{k,l}}{1 - h^k}.
\end{aligned}$$

We assume this new sequence to be independent of $\{\zeta(n)\}$. We now choose:

$$\nu_j^k(n) = \beta_j^k(n)\tilde{\nu}_j^k(n) + (1 - \beta_j^k(n))\bar{\nu}_j^k(n).$$

For $n = 1, 2, \dots$, consider the network $\tilde{\Sigma}_{[1,n]}$ with driving sequence $\tilde{\xi}(i) \equiv \{\zeta(i),\ \{\tilde{\nu}_j^k(i)\}\}$ and let

$${}_V\tilde{\Sigma}_{[1,n]} = V + \tilde{\Sigma}_{[1,n]},$$

where V is supposed to be such that ${}_V W \leq W(0, C)$ a.s. Since non-initial customers have acyclic routes and since the traffic intensity is less than 1, then (see Foss [19]) there exists an a.s. finite, positive integer-valued r.v. ζ such that, for all n

$$\tilde{W}_{[1,n]} = {}_V\tilde{W}_{[1,n]} = {}_{W(0,C)}\tilde{W}_{[1,n]} \quad a.s. \tag{150}$$

on the event $\{n \geq \zeta\}$. Let

$$L = \min\{n \geq 1 : \quad \mathbf{P}(\zeta = n) > 0\}.$$

Then the event

$$A \equiv \{\zeta = L\} \bigcap \{\beta_j^k(i) = 1,\ i = 1, \ldots, L,\ k = 0, \ldots, K,\ j = 1, \ldots, \tilde{\varphi}^k(i)\}$$

has a positive probability. Therefore the events $\{\{\zeta \leq n\} \bigcap \{\theta^n A\}, \quad n = 1, 2, \ldots\}$ form a sequence of renovating events for the sequence $\{{}_V W_{[1,n]}\}$, and the statement of the theorem follows from Theorem 3 of Borovkov and Foss [12] or of Foss [19]. ■

8 Appendix

8.1 Appendix A: The Geometry of Routes

Case $N = 1$

Lemma 11 *Let $r = (r_1, \ldots, r_{\varphi-1}, K+1)$ be a successful route and G be the E.O.D.G. generated by r. Choose $k \in \{1, \ldots, K\}$ such that $d^k > 0$. Consider the path $\tilde{r}$ starting from k, $\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_m)$. Then $\tilde{r}$ is an admissible route.*

Proof The path from node k is also the sequence of nodes produced by Procedure 2 with input (G, k) (because G has a single route); we will refer to an object associated with G_t^k in this procedure by adding the subscript t. For all t and $l \in \{1, \ldots, K\}$, $c_t^l = d_t^l$ (see Remark 6). Assume that the path originating from k ends in node $l \neq K + 1$. This means that there exists an integer t such that $d_t^l = 1$ and $c_t^l = 1$, while $d_{t+1}^l = 0$ and $c_{t+1}^l = 0$, which is only possible via a k-reduction if $l = k$. ■

Lemma 11 shows that there are actually only two types of paths: circuits and routes ending in $K + 1$. We shall study their structure in each case, in Lemmas 12 to 15 below. In what follows, we will denote $\hat{G}^k$ the k-sequential residual of G (the k superscript will be omitted when non-ambiguous) .

Lemma 12 *If the path $\tilde{r}$ of Lemma 11 is a simple circuit of length m (i.e. $\tilde{r}_i \neq \tilde{r}_j$ for all $1 \leq i, j < m$, $i \neq j$ and $\tilde{r}_m = \tilde{r}_1$), then we can find $l \in \{1, \ldots, K\}$ such that $\tilde{r}_i = l$ for some $i \in \{1, \ldots, m-1\}$ (i.e. l belongs to the path $\tilde{r}$), and such that the route r can be represented under the form:*

$$r = (r_1, \ldots, r_a, \ldots, r_b, \ldots, r_\varphi, K + 1)$$

where $r_a = r_b = l$ and $(r_a, \ldots, r_b)$ is a cyclic permutation of $\tilde{r}$:

$$(r_a, \ldots, r_b) = (\tilde{r}_i, \tilde{r}_{i+1}, \ldots, \tilde{r}_{m-1}, \tilde{r}_1, \ldots, \tilde{r}_{i-1}, \tilde{r}_i).$$

Proof Let A be the set of nodes in the sequence $\{\tilde{r}_i\}_{i=1}^{m-1}$. Let $a = \min\{n : r_n \in A\}$ and $l = r_a = \tilde{r}_i$. ¿From the definition of a path, $r_{a+j} = \tilde{r}_{i+j}$, for $0 \leq j \leq m-1-i$, and $r_{a+j} = \tilde{r}_{1+i-m}$ for $m-i \leq j \leq m-1$. ■

Corollary 19 *The k-sequential residual $\hat{G}$ is an E.O.D.G. generated by the successful route*

$$\hat{r} = (r_1, \ldots, r_a, r_{b+1}, \ldots, r_{\varphi-1}, K+1).$$

Lemma 13 *If the path $\tilde{r}$ in Lemma 11 is such that $\tilde{r}_i \neq \tilde{r}_j$ for all $1 \leq i, j, < m$ and $\tilde{r}_m = K+1$, then we can find an integer a such that*

$$\tilde{r} = (r_a, r_{a+1}, \ldots, r_{\varphi-1}, K+1).$$

In addition, the k-sequential residual $\hat{G}$ is an O.D.G. generated by the route

$$\hat{r} = (r_1, \ldots, r_a),$$

which is a circuit.

Proof Define A, a and l as in the proof of Lemma 12. From the definition of the path, $r_{a+j} = \tilde{r}_{i+j}$ for $0 \leq j \leq m-i$. In particular, $r_{a+m-i} = \tilde{r}_m = K+1$. But $\varphi^k > 0$. So $l = k$ and $\tilde{r}_1 = r_a$. The fact that $\hat{r}$ is a circuit follows from Lemma 11. ■

Lemma 14 *If $\tilde{r}$ is a general circuit (i.e. a path such that $\tilde{r}_m = \tilde{r}_1$), then we can find finite integers $l \geq 1$ and*

$$a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_l < b_l < \varphi$$

such that $r_{a_i} = r_{b_i}$ and such that the associated k-sequential residual $\hat{G}$ is an E.O.D.G. generated by the successful route

$$\hat{r} \equiv (r_1, \ldots, r_{a_1}, r_{b_1+1}, \ldots, r_{a_2}, r_{b_2+1}, \ldots, r_{a_l}, r_{b_l+1}, \ldots, r_{\varphi-1}, K+1). \tag{151}$$

Proof The proof is by induction on the length m of the path. For $m = 2$, the statement is clear, because circuits of length 2 are necessarily simple. Assume we proved the property for all m_0 with $1 \leq m_0 < m$. If $\tilde{r}$ is a simple circuit, then the statement is true from Lemma 12 and Corollary 19. Otherwise let α and β be the smallest integers such that $1 \leq \alpha < \beta$ and such that $(\tilde{r}_\alpha, \tilde{r}_{\alpha+1}, \ldots, \tilde{r}_\beta)$ is a simple circuit; the circuit $\tilde{r}' \equiv (\tilde{r}_\alpha, \tilde{r}_{\alpha+1}, \ldots, \tilde{r}_\beta)$ is necessarily the prefix of a path of G. Let $\hat{G}'$ be the $(\tilde{r}_\alpha, (\beta - \alpha + 1))$-sequential residual of G. Using the same arguments as in Lemma 12, we obtain that $\hat{G}'$ is an E.O.D.G. generated by a route of the form

$$\hat{r}' = (r_1, \ldots, r_a, r_{b+1}, \ldots, r_{\varphi-1}, K+1) \equiv (r'_1, \ldots, r'_{\varphi'-1}, K+1), \tag{152}$$

where $(r_a, \ldots, r_b)$ is a cyclic permutation of $(\tilde{r}_\alpha, \ldots, \tilde{r}_\beta)$ and $r_a = r_b$. Corollary 19 also implies that $\tilde{r}'' \equiv (\tilde{r}_1, \ldots, \tilde{r}_\alpha, \tilde{r}_{\beta+1}, \ldots, \tilde{r}_m)$ is a path of length $m'' < m$ for the E.O.D.G. $\hat{G}'$. ¿From the induction assumption, the $\tilde{r}_1$-sequential residual of $\hat{G}'$ is an E.O.D.G. generated by a route $\hat{r}''$ of the form given in (151), namely.

$$\hat{r}'' \equiv (r'_1, \ldots, r'_{a'_1}, r'_{b'_1+1}, \ldots, r'_{a'_2}, r'_{b'_2+1}, \ldots, r'_{a'_l}, r'_{b'_l+1}, \ldots, r'_{\varphi'-1}, K+1), \tag{153}$$

where $r'_{a'_i} = r'_{b'_i}$ for all i. Equations (152)-(153) show that the property holds for all circuits of length m. ■

Lemma 15 *If $\tilde{r}$ is such that $\tilde{r}_m = K+1$, then we can find finite integers $l \geq 1$ and a_i, b_i, $i = 1, \ldots, l$ such that*

$$r_{a_i} = r_{b_i} \quad \forall i = 1, \ldots, l-1,$$

and such that the residual graph $\hat{G}$ is an O.D.G. generated by the route

$$\hat{r} \equiv (r_1, \ldots, r_{a_1}, r_{b_1+1}, \ldots, r_{a_2}, r_{b_2+1}, \ldots, r_{a_l}).$$

In general, this route is not admissible.

Proof The proof follows from Lemmas 13-14 and from induction arguments. ■

Case $N = 2$

Lemma 16 *Theorem 3 holds true for $N = 2$.*

Proof Let G be an E.O.D.G. with a two route generator $R = (r(1), r(2))$, where $r_2(1) = l_1$ and $r_2(2) = l_2$. Let G' be the $0, \sigma$-permutation of G, with $\sigma(1) = 2$, $\sigma(2) = 1$. We show that G' is an E.O.D.G. by constructing the two routes $\tilde{r}(1)$ and $\tilde{r}(2)$ which generate G'. If $l_1 = l_2$, then we take $\tilde{r}(1) = r(1)$ and $\tilde{r}(2) = r(2)$. Otherwise let $L = (L_1, \ldots, L_m)$ be the path of G originating from node $L_1 = l_2$. Let d'^k and c'^k count the number of arcs from and to node $k = 1, \ldots, K+1$, excluding those coming from node 0. We have $d'^k = c'^k$ for all $k \notin \{l_1, l_2\}$, and $d'^k = c'^k + 1$ for $k \in \{l_1, l_2\}$; so $L_m = K+1$ necessarily. We take $\tilde{r}(1) = L$. In order to define $\tilde{r}(2)$, consider the last arc (L_{m-1}, L_m) of L. There are two possibilities:

- **(a) This arc belongs to the route** $r(2)$**.**
  If the arc (L_1, L_2) belongs to $r(1)$, Lemma 12 implies that we can find an integer $q \geq 2$ such that $L_q = L_1$ and all the arcs $(L_1, L_2), (L_2, L_3), \ldots, (L_{q-1}, L_q)$ belong to $r(1)$, and (L_q, L_{q+1}) belongs to $r(2)$.
  If for some $p \geq 2$, (L_{p-1}, L_p) belongs to $r(2)$ and (L_p, L_{p+1}) belongs to $r(1)$, then we can also find an integer $q \geq p+1$ such that the arcs $(L_p, L_{p+1}), (L_{p+1}, L_{p+2}), \ldots, (L_{q-1}, L_q)$ all belong to $r(1)$ and $L_p = L_q$, while (L_q, L_{q+1}) belongs to $r(2)$.

  So, whatever the initial arc of L, the sequence of arcs of path L is composed of circuits which all belong to $r(1)$ and of certain sequences of arcs belonging to $r(2)$.

  We now prove that in fact *all* the arcs belonging to $r(2)$ also belong to L. Consider the first arc of $r(2)$ (remember that this arc originates in L_1). If it does not belong to the sequence of arcs of L, what precedes implies that the path L has to return to L_1 infinitely often; since

$\varphi(1)$ is finite, this is not possible. Thus, the first arc of $r(2)$ belongs to L. The same argument is applicable (by induction) to all arcs belonging to $r(2)$.

Let $\hat{G}$ be the 0-sequential residual of G'. Thus the set of arcs of $\hat{G}$ consists of all arcs of route $r(1)$ but for a finite number of circuits. It follows from Lemma 14 and from an immediate induction argument that $\hat{G}$ is an E.O.D.G. generated by some successful route $\tilde{r}(2)$ originating from node l_1.

- **(b) This arc belongs to route $r(1)$.**
  If the path L only consists of arcs which belong to $r(1)$, then L is a path from l_2 to $K+1$, and we prove from Lemma 15 that the 0-sequential residual of G' is then a successful route $\tilde{r}(2)$ made of the concatenation of two paths: (i) a set of arcs, all belonging to $r(1)$, starting with nodes 0 and l_1, and ending in node l_2; (ii) route $r(2)$.

  Assume now that there exists an integer $q \geq 2$ such that (L_{q-1}, L_q) belongs to $r(2)$ and (L_{q+i}, L_{q+i+1}) to $r(1)$, for all $i \geq 0; i < m - q$. For the same reasons as in (a), the arcs (L_{j-1}, L_j), $j \leq q$, which belong to $r(1)$, form a finite number of circuits. We prove as above that the set of the arcs which belong to $r(1)$ and not to L is generated by some route $\check{r}(1) \equiv (\check{r}_1(1), \ldots, \check{r}_n(1))$, the form of which is given by Lemma 15. We have in particular $\check{r}_1(1) = l_1$ and $\check{r}_n(1) = L_q$.

  Concerning the arcs of L which belong to $r(2)$, as in case (a), simple induction arguments show that we can find a number p such that all the arcs $(r_1(2), r_2(2)), \ldots, (r_{p-1}(2), r_p(2))$ (and only these arcs of $r(2)$) belong to L. Moreover, $r_p(2)$ has to be equal to L_q.

  We take $\tilde{r}(2) = (\check{r}_1(1), \ldots, \check{r}_n(1), r_{p+1}(2), \ldots, r_{\varphi(2)-1}(2), K+1)$.

■

Proof of Theorem 3

Fix $n \in \{1, \ldots, N\}$. We shall show that if there exists a generator $R = (r(1), \ldots, r(N))$ for G, then there exists a generator $\tilde{R} = (\tilde{r}(1), \ldots, \tilde{r}(N))$ for the $(0, \sigma)$-permutation of G, where $\sigma(1) = n$. The statement of Theorem 3 follows by induction. We define $\tilde{R}$ as follows:

- take $\tilde{r}(j) = r(j)$ for all $j \neq n, j \neq n-1$;
- consider the E.O.D.G. G' with generator $(r(n-1), r(n))$ and use Lemma 16.

■

8.2 Appendix B: More on First-Order Ergodic Theorems

For $j = -F^k(-n) + 1, \ldots, 0$ let

$$\psi^k_{j,[-n,0]} \tag{154}$$

be the epoch of the completion of the j-th service on station k in $\Sigma_{[-n,0]}$ and let

$$Z^k_{[-n,0]} = \psi^k_{0,[-n,0]} - t(0) \tag{155}$$

be the moment of the completion of the last service on station k in $\Sigma_{[-n,0]}$ (with the convention that = $-\infty$ if $F^k_{[-n,0]} = 0$); similarly, for $l \le n$, let

$$Z^k_{-l,[-n,0]} = \psi^k_{-F^k_{-l},[-n,0]} - t(0) \tag{156}$$

(= $-\infty$ if $F^k(-n) = F^k(-l)$),

$$Z_{-l,[-n,0]} = \max_{1\le k\le K} Z^k_{-l,[-n,0]} \tag{157}$$

and

$$S_{[m,n]} = \sum_{j=m}^{n} S(j). \tag{158}$$

Note that

$$Z_{[-n,0]} = \max_{1\le k\le K} (Z^k_{[-n,0]})^+$$

and that

$$(Z_{-l,[-n,0]})^+ \le Z_{[-n,0]}.$$

Lemma 17 *For all* $0 \le l \le n$

$$Z_{[-n,0]} \le (Z_{-l,[-n,0]})^+ + S_{[-l,0]} \tag{159}$$

Proof The case $Z_{[-n,0]} = 0$ is trivial. Assume that $Z_{[-n,0]} > 0$. At each instant of the time interval $(0, Z_{[-n,0]})$ at least one customer is being served. In addition, from time $(Z_{-l,[-n,0]})^+$ on, the services completed on station k have an index larger than $(-F^k(-l) + 1)$, for all $k = 1\ldots, K$. Since $(Z_{-l,[-n,0]})^+ \ge 0$, $Z_{[-n,0]} - (Z_{-l,[-n,0]})^+$ is bounded from above by $S_{[-l,0]}$. ■

Corollary 20 *If* $\mathbf{P}(A) = 1$ *then, for all fixed* $l \le n$*,* $Z_{-l,[-n,0]} \to \infty$ *a.s. as* n *tends to* ∞*.*

Proof Let $\beta = \min\{n \ge 0 : Z_{-l,[-n,0]} \ge 0\}$. Then $(Z_{[-n,0]} - Z_{-l,[-n,0]})I(n \ge \beta) \le S_{[-l,0]}$ a.s.. In particular,

$$\frac{Z_{[-n,0]} - Z_{-l,[-n,0]}}{g(n)} \to 0 \tag{160}$$

a.s. as $n \to \infty$ for any function $g(n)$ such that $\lim g(n) = \infty$. ■

Corollary 21 *Let* $l = l(n)$ *be some non-decreasing (possibly random) integer-valued function of* n *such that* $l(n)/g(n) \to 0$ *(a.s.) as* $n \to \infty$*, for* g *as above. Then*

$$(Z_{[-n,0]} - Z_{-l(n),[-n,0]})I(Z_{-l(n),[-n,0]} \ge 0)I(l(n) \le n)/g(n) \to 0 \tag{161}$$

a.s. as n *tends to* ∞*. In particular, if* $l(n) = l$ *with* l *random, then* $\mathbf{P}(A) = 1$ *implies that* $Z_{-l,[-n,0]} \to \infty$ *a.s. and in addition,*

$$\limsup(Z_{[-n,0]} - Z_{-l,[-n,0]}) \le S_{[-l,0]} \tag{162}$$

a.s. as n *tends to* ∞*.*

Proof If $l(\infty) \equiv \lim_n l(n)$ is finite, then the result follows from Corollary 20 and from monotonicity properties. If $l(\infty) = \infty$, then write

$$\frac{S_{[-l(n),0]}}{g(n)} = \frac{S_{[-l(n),0]}/l(n)}{g(n)/l(n)} \tag{163}$$

and since

$$S_{[-l(n),0]}/l(n) \to \mathbf{E}S(0)$$

a.s. on the event $\{l(\infty) = \infty\}$, we have

$$\begin{aligned} 0 &\le (Z_{[-n,0]} - Z_{[-n,-l(n),0]})I(Z_{[-n,-l(n),0]} \ge 0)I(l(n) \le n) \\ &\le \frac{(S_{[-l(n),0]}/l(n))}{g(n)/l(n))} I(l(n) \le n) \to 0 \end{aligned}$$

a.s. as n tends to ∞. ■

Assume now that

$$\mathbf{E}\varphi^{k,i}(0) > 0 \tag{164}$$

for all $1 \le k, i \le K$. This property should be understood as some strong connectedness property of the routing mechanism. For $m \ge 0$, let

$$\beta^{i,k}(m) = \min\{j > m : \varphi^{i,k}(-j) > 0\} \tag{165}$$

and

$$\beta^k(m) = \max_i \beta^{i,k}(m). \tag{166}$$

Corollary 22 (Solidarity property) *Under the condition (164), for each* $1 \le k \le K$, $n \ge m \ge 0$

$$(Z_{[-n,0]} - Z^k_{-m,[-n,0]})I(\beta^k(m) \le n)I(Z_{-\beta^k(m),[-n,0]} \ge 0) \le S_{[-\beta^k(m),0]} \tag{167}$$

a.s. In particular, if $\mathbf{P}(A) = 1$ *then*

$$\limsup(Z_{[-n,0]} - Z^k_{-m,[-n,0]}) \le S_{[-\beta^k(m),0]} \tag{168}$$

a.s. as $n \to \infty$ *for all* $m \ge 0$ *and* $k = 1, \ldots, K$ *and*

$$(Z_{[-n,0]} - Z^k_{-m,[-n,0]})/g(n) \to 0 \tag{169}$$

for all $g(n)$ *such that* $g(n) \to \infty$. *In particular,*

$$\lim_n Z^k_{-m,[-n,0]}/n = \gamma \quad a.s. \tag{170}$$

for all k *and* m, *where* γ *is the constant defined in Theorem 11.*

Proof The result follows from the inequality

$$(Z_{-\beta^k(m),[-n,0]} - Z^k_{-m,[-n,0]})I(\beta^k(m) \le n) \le 0.$$

■

8.3 Appendix C

Proof of

$$\frac{Y_{(m_n+1,n)}}{n-m_n} \to \gamma(0) \quad \text{in probability.}$$

For $0 < \delta < c$, let

$$x_n = [(c-\delta)n]; \quad y_n = [(c+\delta)n] + 1,$$

where $[x]$ denotes the integer part of x. Let $H_n = Y_{(m_n+1,n)}/(n-m_n) - \gamma(0)$. For all $\epsilon > 0$

$$\mathbf{P}(|H_n| > \epsilon) = \mathbf{P}(H_n > \epsilon) + \mathbf{P}(H_n < -\epsilon)$$

and for all $0 < \delta < \epsilon(1-c)/(2\gamma(0)+\epsilon)$

$$\mathbf{P}(H_n > \epsilon) \le \mathbf{P}(|\frac{m_n}{n} - c| > \delta) + \mathbf{P}(\frac{Y_{(x_n,n)}}{n-y_n} > \gamma(0)+\epsilon).$$

The last expression tends to 0 as n goes to ∞, as it can be seen from the following relations:

$$\begin{aligned} \frac{Y_{(0,n-x_n)}}{n-x_n} &\to \gamma(0); \\ \frac{n-x_n}{n-y_n} &\to \frac{1-c+\delta}{1-c-\delta}; \\ \gamma(0)\frac{1-c+\delta}{1-c-\delta} &= \gamma(0)\frac{1+2\delta}{1-c-\delta} < \gamma(0)(1+\frac{\epsilon}{\gamma(0)}) = \gamma(0)+\epsilon. \end{aligned}$$

$\mathbf{P}(H_n < -\epsilon) \to 0$, by similar arguments. ■

Proof of

$$\tilde{Z}_{(1,m_n)}/n \to 0 \quad \text{in probability.}$$

Note that $\tilde{Z}_{(1,m)} \le \tilde{Z}(m)$ a.s. for each m (see (110) and

$$\tilde{Z}_{(1,m+l)} \le \tilde{Z}_{(1,m)} + \tilde{Z}_{(m+1,m+l)} \le \tilde{Z}(m) + \sum\nolimits_{j=m+1}^{m+l} \tilde{Z}_{(j)},$$

where $\tilde{Z}_{(j)}$ is a stationary and ergodic sequence with finite first moment $\mathbf{E}\tilde{Z}_{(1)} \equiv h$.

Therefore for all $\epsilon > 0$, $0 < \delta < \min(c, \epsilon/4h)$

$$\begin{aligned} \mathbf{P}(\tilde{Z}_{(1,m_n)}/n > \epsilon) &\le \mathbf{P}(|m_n/n - c| > \delta) + \mathbf{P}(\max\nolimits_{x_n \le l \le y_n} \tilde{Z}_{(1,l)}/n > \epsilon) \\ &\le \mathbf{P}(|m_n/n - c| > \delta) + \mathbf{P}(\{\tilde{Z}(x_n) + \sum\nolimits_{j=x_n+1}^{y_n} \tilde{Z}_{(j)}\}/n > \epsilon) \\ &\le \mathbf{P}(|m_n/n - c| > \delta) + \mathbf{P}(\tilde{Z}(x_n) > \epsilon/2) + \mathbf{P}(\sum\nolimits_{j=1}^{y_n-x_n} \tilde{Z}_{(j)} > \epsilon/2). \end{aligned}$$

The last expression tends to 0 as n goes to ∞ because

$$\begin{aligned} \mathbf{P}(\tilde{Z}(x_n) > \epsilon/2) = \mathbf{P}(\tilde{Z}(1) > n\epsilon/2) &\to 0; \\ (\sum\nolimits_{j=1}^{y_n-x_n} \tilde{Z}_{(j)})/(y_n - x_n) &\to h; \\ \frac{y_n - x_n}{n} &\to 2\delta < \epsilon/2h. \end{aligned}$$

■

Future Work

The consequences of the construction that is proposed here will be investigated in a companion paper which will primarily focus on the i.i.d case. We would also like to mention that these techniques extend almost directly to the class of Petri nets defined in Baccelli, Cohen, Olsder and Quadrat [5] (see Baccelli and Foss [7]).

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