

Discrete and continuous time modulated random walks with heavy-tailed increments

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Abstract

We consider a modulated process S which, conditional on a background process X , has independent increments. Assuming that S drifts to $-\infty$ and that its increments (jumps) are heavy-tailed (in a sense made precise in the paper), we exhibit natural conditions under which the asymptotics of the tail distribution of the overall maximum of S can be computed. We present results in discrete and in continuous time. In particular, in the absence of modulation, the process S in continuous time reduces to a Lévy process with heavy-tailed Lévy measure. A central point of the paper is that we make full use of the so-called “principle of a single big jump” in order to obtain both upper and lower bounds. Thus, the proofs are entirely probabilistic. The paper is motivated by queueing and Lévy stochastic networks.

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1 Introduction

This paper deals with the study of the asymptotic distribution of the maximum of a random walk S on the real line \mathbb{R} , modulated by a regenerative process, when the increments have heavy-tailed distributions. (By “modulated” we mean that, conditional on some background process, S becomes a process with independent increments.)

In the case where the modulating process is a finite-state Markov chain, and hence has regeneration intervals whose distribution is exponentially bounded, the asymptotic behaviour of the maximum of the random walk is well-understood (see below). For a more general modulating process only limited results are available; in particular there is little concerning the relation between the tail of the distribution of the regenerative intervals and that of the maximum of the random walk.

The present paper has two goals. The first is to introduce new techniques for the study of this problem. These techniques generalise and unify existing results, and, for a general (regenerative) modulating process, yield a very precise answer as to the conditions under which the hoped-for results may be obtained. Further the approach is entirely probabilistic, and depends in particular on what we may call the *principle of a single big jump*, namely the folklore fact that achieving a high value of the maximum of the random walk is essentially due to a single very large jump. Indeed, we manage to translate this principle into rigorous statements that make the proofs quite transparent. In doing so we formulate and prove

results of independent interest for processes with independent, heavy-tailed, increments. Our second goal is to obtain the corresponding results for continuous-time modulated random walks. Again we introduce new probabilistic techniques for this problem. These are closely related to those for the discrete-time case: there are additional considerations arising from the diffusion and “small jumps” components of what is in effect a modulated Lévy process, but nevertheless some arguments are simpler in this case.

Throughout the paper, by “tail” we mean exclusively “right tail”, except where otherwise explicitly stated. By a *heavy-tailed* distribution we mean a distribution (function) G on \mathbb{R} possessing no exponential moments: $\int_0^\infty e^{sy} G(dy) = \infty$ for all $s > 0$. Such distributions not only abound in theory, but are also useful in modern probabilistic modelling, in such diverse areas as insurance risk, communication networks, and finance.

For any distribution function G on \mathbb{R} , we set $\overline{G}(y) := 1 - G(y)$ and denote by G^{*n} the n -fold convolution of G by itself.

A distribution G on \mathbb{R}_+ belongs to the class \mathcal{S} of *subexponential* distributions if and only if, for all $n \geq 2$, we have $\lim_{y \rightarrow \infty} \overline{G^{*n}}(y)/\overline{G}(y) = n$. (It is sufficient to verify this condition in the case $n = 2$ —see Chistyakov (1964).) This statement is easily shown to be equivalent to the condition that, if Y_1, \dots, Y_n are i.i.d. random variables with common distribution G , then

$$P(Y_1 + \dots + Y_n > y) \sim P(\max(Y_1, \dots, Y_n) > y),$$

a statement which already exemplifies the principle of a single big jump. (Here, and elsewhere, for any two functions f, g on \mathbb{R} , by $f(y) \sim g(y)$ as $y \rightarrow \infty$ we mean $\lim_{y \rightarrow \infty} f(y)/g(y) = 1$; we also say that f and g are *tail-equivalent*. We also write $f(y) \lesssim g(y)$ as $y \rightarrow \infty$ if $\lim_{y \rightarrow \infty} f(y)/g(y) \leq 1$.) The class \mathcal{S} includes all the heavy-tailed distributions commonly found in applications, in particular regularly-varying, lognormal and Weibull distributions.

If G_1 and G_2 are distributions on \mathbb{R}_+ such that $G_1 \in \mathcal{S}$ and $\overline{G_2}(y) \sim c\overline{G_1}(y)$ as $y \rightarrow \infty$ for some constant $c > 0$, then also $G_2 \in \mathcal{S}$ —see Klüppelberg (1988). In particular, subexponentiality is a tail property, a result of which we make repeated implicit use below. It is thus natural to extend the definition of subexponentiality to distributions G on the entire real line \mathbb{R} by defining $G \in \mathcal{S}$ if and only if $G_+ \in \mathcal{S}$ where $G_+(y) = G(y)\mathbf{1}(y \geq 0)$ and $\mathbf{1}$ is the indicator function. Some further necessary results for subexponential distributions are given in the Appendix.

We define also the class \mathcal{L} of *long-tailed* distributions on \mathbb{R} by $G \in \mathcal{L}$ if and only if, for all c , $\overline{G}(y+c) \sim \overline{G}(y)$ as $y \rightarrow \infty$ (it is sufficient to verify this for any nonzero c). It is known that $\mathcal{S} \subset \mathcal{L}$ and that any distribution in \mathcal{L} is heavy-tailed—see Embrechts and Omey (1982). Good surveys of the basic properties of heavy-tailed distributions, in particular long-tailed and subexponential distributions, may be found in Embrechts *et al.* (1997) and in Asmussen (2000).

For any distribution G on \mathbb{R} with finite mean, we define the integrated (or second) tail distribution (function) G^{I} by

$$\overline{G^{\text{I}}}(y) = 1 - G^{\text{I}}(y) := \min \left(1, \int_y^\infty \overline{G}(z) dz \right).$$

Note that $G \in \mathcal{L}$ implies that $G^{\text{I}} \in \mathcal{L}$, but not conversely.

Let $(\xi_n, n \geq 1)$ be a sequence of i.i.d. random variables with common distribution F on \mathbb{R} and define the random walk $(S_n, n \geq 0)$ by $S_n = \sum_{i=1}^n \xi_i$ for each $n \geq 0$ (with the convention here and elsewhere that a summation over an empty set is zero, so that here $S_0 = 0$). Define $M := \sup_{n \geq 0} S_n$. A now classical result (Pakes (1975), Veraverbeke (1977), Embrechts and

Veraverbeke (1982)), which we henceforth refer to as the Pakes-Veraverbeke's Theorem, states that if $F^\dagger \in \mathcal{S}$ and if $a := -E\xi_1 > 0$ (so that in particular M is a.s. finite) then

$$P(M > y) \sim \frac{1}{a} F^\dagger(y) \quad \text{as } y \rightarrow \infty. \quad (1)$$

(Again it is the case that for most common heavy-tailed distributions F , including those examples mentioned above, we have $F^\dagger \in \mathcal{S}$.) The intuitive idea underlying this result is the following: the maximum M will exceed a large value y if the process follows the typical behaviour specified by the law of large numbers, i.e. it's mean path, except that at some one time n a jump occurs of size greater than $y + na$; this has probability $\bar{F}(y + na)$; replacing the sum over all n of these probabilities by an integral yields (1); this again is the principle of a single big jump. See Zachary (2004) for a short proof of (1) based on this idea.

In the first part of the paper (Section 2) we consider a sequence $(\xi_n, n \geq 1)$ of random variables which, conditional on another process $X = (X_n, n \geq 1)$, are independent, and which further are such that the conditional distribution of each ξ_n is a function of X_n and otherwise independent of n . We then talk of the partial sums $S_n := \sum_{i=1}^n \xi_i$ as a *modulated* random walk. (In fact our framework includes a variety of apparently more general processes, e.g. Markov additive processes—see Remark 2.1.) Our aim is to obtain the appropriate generalisation of the result (1), with only the minimum of necessary conditions on the modulating, or background, process X . We give references to earlier work below.

We need to assume some asymptotic stationarity for the background process X which we take to be regenerative. A particular case of this is when X is an ergodic Markov chain. We also suppose that the conditional distributions F_x given by $F_x(y) := P(\xi_n \leq y \mid X_n = x)$ have tails which are bounded above by that of a *reference* distribution F such that $F^\dagger \in \mathcal{S}$. We then show (in Theorem 2.1) that, in the case where the distributions F_x (when truncated sufficiently far below) have means which are uniformly negative, then, under very modest further regularity conditions, the result (1) continues to hold with $1/a$ replaced by C/a . Here $-a < 0$ is now the average (with respect to the stationary distribution of X) of the above means and the constant C measures the average weight of the tail of F_x with respect to that of F . (The condition $-a < 0$ is sufficient to ensure that M is a.s. finite.) In the more general case where the distributions F_x have means of arbitrary sign, but where $-a$ defined as above continues to be negative, we show (in Theorem 2.2) that the result (1) continues to hold, with $1/a$ replaced by C/a as above, provided that an appropriate condition is imposed on the tail of the distribution of the lengths of the regenerative cycles of the process X . We give an example to show the necessity of this condition. Our proofs follow the probabilistic intuition of a single big jump as defined above. One key idea, encapsulated in a very general result given in Section 2.2 and applicable to a wide class of processes with independent heavy-tailed increments, is to use the result (1) of the Pakes-Veraverbeke Theorem itself: the extremes of the increments of the general process may be bounded by those of an unmodulated random walk, whose increments are i.i.d. with negative mean; the fact that an extreme value of the supremum of the latter process may only be obtained via a single big jump ensures the corresponding result for the modulated process. Indeed we only ever use the condition $F^\dagger \in \mathcal{S}$ in the application of the Pakes-Veraverbeke Theorem (though we make frequent use of the weaker condition $F^\dagger \in \mathcal{L}$). A preliminary version of the discrete-time theory was given in Foss and Zachary (2002). The present treatment is considerably simpler and more unified, and results are obtained under weaker conditions which are, in a sense, demonstrated in Example 2.1, optimal.

We mention several papers on the tail asymptotics of a the supremum of a discrete-time modulated random walk with heavy-tailed increments. Arndt (1980) considers increments

with regularly varying tails modulated by a finite-state-space Markov chain. Alsmeyer and Sgibnev (1999) and, independently, Jelenkovic and Lazar (1999) also consider a finite state space Markov chain as the modulating process, and assume that the increments of the modulated process have a subexponential integrated tail. Note that, for a finite Markov chain, the cycle length distribution has an exponential tail. Asmussen, Schmidli and Schmidt (1999) consider a modulated random walk and assume that the distribution of the maximum increment of the process over any cycle has a tail which is not heavier than that of the distribution of the final increment over the cycle. Asmussen (1999) also considers a modulated random walk with an exponentially bounded distribution of the cycle length, and assumes that both the tails and the integrated tails of the increments of the modulated process have subexponential distributions. Asmussen and Møller (1999) and Asmussen (1999) also consider a random walk with another type of modulation.

Hansen and Jensen (2005) study the asymptotics of the maximum of a modulated process on a finite random time horizon. We also mention a number of related papers on queueing systems whose dynamics may be viewed as a kind of multi-dimensional random walk with a special type of modulation. Baccelli, Schlegel, and Schmidt (1999) and Huang and Sigman (1999) consider a special type of modulation which occurs in tandem queues and in their generalisations, and find asymptotic results under the assumption that the tail distribution of one of the service times strictly dominates the remainder. A general approach to the asymptotic study of monotone separable stochastic networks is given by Baccelli and Foss (2004), see also Baccelli, Foss and Lelarge (2004, 2005) for further applications.

In the second part of the paper (Section 3), we consider the supremum of modulated continuous-time random walk, whose jumps are similarly heavy-tailed. The modulated continuous-time random walk is defined as a process $(S_t, t \geq 0)$ which, conditional on a regenerative process $(X_t, t \geq 0)$ has independent increments, i.e. its characteristic function is given by the Lévy-Khinchine formula. The parameters of the process entering the Lévy-Khinchine formula are therefore themselves measurable functions of the background regenerative process. In the absence of the background process, $(S_t, t \geq 0)$ becomes a Lévy process. Under conditions analogous to those for the discrete-time theory, we establish similar results for the asymptotic tail distribution of the supremum of the process $(S_t, t \geq 0)$. The continuous-time theory quite closely parallels the discrete (and we make every attempt to keep the two developments as similar as possible): there are, however, some additional technicalities concerned with the “small jumps” and diffusion components of the continuous-time process—these do not contribute to the heavy-tailed distribution of the supremum; in compensation some aspects of the theory are simpler than in discrete time. In particular, the proof of the lower bound in continuous-time requires the use of a (modulated) Poisson point process in a way that is similar to the methods of Konstantopoulos and Richardson (2002). Again we require a result, given in Section 3.3, for a fairly general class of processes with independent heavy-tailed increments. The specialisation of this result, under appropriate conditions, to an (unmodulated) Lévy process gives a simple proof of the continuous-time version of the Pakes-Veraverbeke Theorem, different from that found in the existing literature—see, e.g., Klüppelberg, Kyprianou and Maller (2004) and Maulik and Zwart (2005).

Some words on motivation: heavy-tailed random variables play a significant role in the mathematical modelling of communication networks because the variety of services offered by a huge system such as the Internet results in heterogeneous traffic. Part of the traffic concerns small requests but other parts pose significant burden to the system resulting in huge delays and queues. Models of networks based on Lévy processes—see, e.g., Konstantopoulos, Last and Lin (2004)—are natural analogues of the more-traditional Brownian networks of

production and service systems. To date, however, no results for the stationary distribution of the load of stations in isolation are available. Our paper represents a first step towards this goal. Indeed, in a Lévy stochastic network of feedforward type, one may see a downstream node as being in the “background” of a previous node. To apply the results of this paper to Lévy stochastic networks is beyond its scope and is left to a future work.

The Appendix gives some results, known and new, for the addition of subexponential random variables, together with some other complements to the main theory. In particular Lemma A.2 extends a well-known result for the sum of independent subexponential random variables (Lemma A.1) to those which have an appropriate conditional independence property, and is of independent interest.

2 Modulated random walk in discrete time

2.1 Introduction and main results

Consider a regenerative process $X = (X_n, n \geq 1)$ such that, for each n , X_n takes values in some measurable space $(\mathcal{X}, \mathcal{X})$. We say that the random walk $(S_n, n \geq 0)$, defined by $S_0 = 0$ and $S_n = \xi_1 + \dots + \xi_n$ for $n \geq 1$, is *modulated* by the process X if

- (i) conditionally on X , the random variables ξ_n , $n \geq 1$, are independent,
- (ii) for some family $(F_x, x \in \mathcal{X})$ of distribution functions such that, for each y , $F_x(y)$ is a measurable function of x , we have

$$P(\xi_n \leq y \mid X) = P(\xi_n \leq y \mid X_n) = F_{X_n}(y) \quad \text{a.s.} \quad (2)$$

Define

$$M_n := \max(S_0, S_1, \dots, S_n), \quad n \geq 0, \quad M := \sup_{n \geq 0} S_n.$$

Under the conditions which we give below, $S_n \rightarrow -\infty$ a.s. as $n \rightarrow \infty$, and the random variable M is then nondefective. We are interested in deriving an asymptotic expression for $P(M > y)$ as $y \rightarrow \infty$.

Remark 2.1. In fact nothing below changes if, in (2), we allow the distribution of ξ_n to depend on the history of the modulating process X between the last regeneration instant prior to time n and the time n itself. This possible relaxation can either be checked directly, or brought within the current structure by suitably redefining the process X . Thus in particular our framework includes Markov additive processes.

The regeneration epochs of the modulating process X are denoted by $0 \leq T_0 < T_1 < \dots$. By definition, the cycles $((X_n, T_{k-1} < n \leq T_k), k \geq 1)$ are i.i.d. and independent of the initial cycle $(X_n, 0 < n \leq T_0)$. Define also $\tau_0 := T_0$ and $\tau_k := T_k - T_{k-1}$ for $k \geq 1$, so that $(\tau_k, k \geq 0)$ are independent and $(\tau_k, k \geq 1)$ are identically distributed. Assume that $E\tau_1 < \infty$. For each $n \geq 0$, let π_n be the distribution of X_n , and define, as usual, the stationary probability measure

$$\pi(B) := \frac{E \sum_{n=T_0+1}^{T_1} \mathbf{1}(X_n \in B)}{E\tau_1}, \quad B \in \mathcal{B}(\mathcal{X}).$$

Each distribution F_x , $x \in \mathcal{X}$, will be assumed to have a finite mean

$$a_x := \int_{\mathbb{R}} y F_x(dy). \quad (3)$$

The family of such distributions will be assumed to satisfy the following additional conditions with respect to some *reference* distribution F with finite mean and some measurable function $c : \mathcal{X} \rightarrow [0, 1]$:

- (D1) $\overline{F_x}(y) \leq \overline{F}(y)$, for all $y \in \mathbb{R}$, $x \in \mathcal{X}$,
- (D2) $\overline{F_x^\Gamma}(y) \sim c(x)\overline{F^\Gamma}(y)$ as $y \rightarrow \infty$, $x \in \mathcal{X}$,
- (D3) $a := -\int_{\mathcal{X}} a_x \pi(dx)$ is finite and strictly positive.

Remark 2.2. The condition (D1) is no less restrictive than the condition

$$\overline{\lim}_{y \rightarrow \infty} \sup_{x \in \mathcal{X}} \frac{\overline{F_x}(y)}{\overline{F}(y)} < \infty,$$

in which case it is straightforward to redefine F , and then c , so that (D1) and (D2) hold as above. Some relaxation of the condition (D1) is made by Shneer (2004) at the expense of requiring substantial further moment conditions on the distributions of the random variables τ_k , $k \geq 0$, defined above.

Remark 2.3. A sufficient condition for (D2) to hold is that, for all $x \in \mathcal{X}$, we have $\overline{F_x}(y) \sim c(x)\overline{F}(y)$ as $y \rightarrow \infty$. However, in order to obtain our main results we shall require Lemma 2.3 below to be established under the weaker condition (D2) as stated above. (The proof of Theorem 2.2 utilises the fact that, when $F^\Gamma \in \mathcal{L}$ as required there, the condition (D2) is preserved when any of the distributions F_x is shifted by a constant. This is not true if (D2) is replaced by the strengthened version above, unless we further assume $F \in \mathcal{L}$ —an assumption which we do not wish to make!)

Remark 2.4. We impose no *a priori* restrictions on the signs of the a_x , other than that given by the condition (D3). The latter condition is trivially satisfied in the case where all the a_x are strictly negative. (The introduction of the minus sign in the definition of a is for convenience in the statement of our results.)

It follows from the regenerative structure of X and from (D1) and (D3), that

$$S_n/n \rightarrow -a \quad \text{as } n \rightarrow \infty, \quad \text{a.s.} \quad (4)$$

(See the Appendix for a proof of this result.)

Thus, in particular, $S_n \rightarrow -\infty$ as $n \rightarrow \infty$ and M is nondefective as required.

For each $x \in \mathcal{X}$ and $\beta > 0$, define

$$a_x^\beta := a_x - \int_{-\infty}^{-\beta} (y + \beta) F_x(dy) = \int_{\mathbb{R}} (y \vee -\beta) F_x(dy); \quad (5)$$

note that $a_x^\beta \geq a_x$. Define also

$$\kappa := \limsup_{\beta \rightarrow \infty} \sup_{x \in \mathcal{X}} a_x^\beta. \quad (6)$$

Note that, from (3) and the condition (D1), κ is a real number between $-a$ and μ , where μ is the mean of the reference distribution F . In the case where the distributions F_x , $x \in \mathcal{X}$, satisfy the uniform integrability condition

$$\limsup_{\beta \rightarrow \infty} \sup_{x \in \mathcal{X}} \int_{-\infty}^{-\beta} |y| F_x(dy) = 0,$$

we have $\kappa = \sup_{x \in \mathcal{X}} a_x$.

Define also $C \in [0, 1]$ by

$$C := \int_{\mathcal{X}} c(x)\pi(dx). \quad (7)$$

We remark that for our results to provide a useful extension to those for the unmodulated case we need $C > 0$; this requires $c(x) > 0$ on some subset $\mathcal{X}' \subseteq \mathcal{X}$ such that $\pi(\mathcal{X}') > 0$, implying essentially the same tail behaviour for F_x , $x \in \mathcal{X}'$.

Theorem 2.1 below gives our main result in the case $\kappa < 0$.

Theorem 2.1. *Suppose that (D1)–(D3) hold, that $F^\dagger \in \mathcal{S}$, and that $\kappa < 0$. Then*

$$\lim_{y \rightarrow \infty} \frac{P(M > y)}{\overline{F^\dagger}(y)} = \frac{C}{a}.$$

In order to extend Theorem 2.1 to the case where the sign of κ may be arbitrary, we require an additional condition regarding the (tail) distributions of the lengths of the regenerative cycles. The condition we need is:

(D4) For some nonnegative $b > \kappa$,

$$P(b\tau_0 > n) = o(\overline{F^\dagger}(n)), \quad P(b\tau_1 > n) = o(\overline{F}(n)), \quad \text{as } n \rightarrow \infty. \quad (8)$$

Note that if (8) is satisfied for some nonnegative b , then it is also satisfied for any smaller value of b . In the case $\kappa < 0$ the condition (D4) is always trivially satisfied by taking $b = 0$. Hence Theorem 2.1 is actually a special case of the general result given by Theorem 2.2 below.

Theorem 2.2. *Suppose that (D1)–(D4) hold and that $F^\dagger \in \mathcal{S}$. Then*

$$\lim_{y \rightarrow \infty} \frac{P(M > y)}{\overline{F^\dagger}(y)} = \frac{C}{a}.$$

In Section 2.4 we give an example to show the necessity of the assumption (D4).

2.2 A uniform upper bound for discrete-time processes with independent increments

Our proofs require several uses of the following proposition, which is new and may be of independent interest. This, under appropriate conditions, provides an upper bound for the distribution of the supremum of a random walk with independent increments. This bound is not simply asymptotic and further has an important uniformity property. No regenerative structure is assumed, and the result is therefore of independent interest.

Proposition 2.1. *Let F be a distribution function on \mathbb{R} such that $\int_0^\infty \overline{F}(y) dy < \infty$ and whose integrated tail $F^\dagger \in \mathcal{S}$. Let α, β be given positive real numbers. Consider any sequence $(\xi_n, n \geq 1)$ of independent random variables such that, for each n , the distribution F_n of ξ_n satisfies the conditions*

$$\overline{F_n}(y) \leq \overline{F}(y) \quad \text{for all } y \in \mathbb{R}, \quad (9)$$

$$\int_{\mathbb{R}} (z \vee -\beta) F_n(dz) \leq -\alpha. \quad (10)$$

Let $M := \sup_{n \geq 0} \sum_{i=1}^n \xi_i$. Then there exists a constant r depending on F , α and β only, such that, for all sequences $(\xi_n, n \geq 1)$ as above,

$$P(M > y) \leq r\overline{F^\dagger}(y) \quad \text{for all } y. \quad (11)$$

Proof. Consider any sequence $(\xi_n, n \geq 1)$ as above. We assume, without loss of generality, that $\xi_n \geq -\beta$, a.s. for all n (for, otherwise, we can replace each ξ_n by $\max(\xi_n, -\beta)$). We now use a coupling construction. Let $(U_n, n \geq 1)$ be a sequence of i.i.d. random variables with uniform distribution on the unit interval $(0, 1)$. For each n , let $F_n^{-1}(y) = \sup\{z : F_n(z) \leq y\}$ be the generalised inverse of F_n , and define similarly F^{-1} . Let

$$\xi_n := F_n^{-1}(U_n), \quad \eta_n := F^{-1}(U_n).$$

Then ξ_n has distribution F_n , η_n has distribution F and $\xi_n \leq \eta_n$ a.s. Choose a constant y^* sufficiently large, such that

$$m := E[\eta_1 \mathbf{1}(\eta_1 > y^*)] \leq \alpha/4 \quad \text{and} \quad \max(1, \beta)P(\eta_1 > y^*) \leq \alpha/4. \quad (12)$$

Let $\varepsilon = P(\eta_1 > y^*)$ and let $K_0 = m/\varepsilon + 1$. For each n , define the random variables

$$\delta_n := \mathbf{1}(\eta_n > y^*) \quad (13)$$

$$\varphi_n := \xi_n(1 - \delta_n) + K_0\delta_n \quad (14)$$

$$\psi_n := (\eta_n - K_0)\delta_n. \quad (15)$$

Note that, from (10), (12)–(15), and our assumption that $\xi_n \geq -\beta$, a.s.,

$$E\varphi_n \leq E\xi_n + (\beta + K_0)E\delta_n \leq -\alpha + (\beta + 1)\varepsilon + m \leq -\alpha/4 \quad (16)$$

and

$$E\psi_n = m - K_0\varepsilon = -\varepsilon < 0. \quad (17)$$

Note also that $(\delta_n, n \geq 1)$ and $(\psi_n, n \geq 1)$ are both sequences of i.i.d. random variables. For each $n \geq 0$, define $S_n^\varphi := \sum_{i=1}^n \varphi_i$ and similarly $S_n^\psi := \sum_{i=1}^n \psi_i$. Define also $M^\varphi := \sup_{n \geq 0} S_n^\varphi$ and $M^\psi := \sup_{n \geq 0} S_n^\psi$. (It will follow below that M^φ and M^ψ are almost surely finite.) From (14), (15), and since $\xi_n \leq \eta_n$ a.s., it follows that, for each n , $\xi_n \leq \varphi_n + \psi_n$, and so

$$M \leq \sup_{n \geq 0} (S_n^\varphi + S_n^\psi) \leq M^\varphi + M^\psi. \quad (18)$$

Given any realisation of the two sequences $(\delta_n, n \geq 1)$ and $(\varphi_n, n \geq 1)$ such that $\sum_n \delta_n = \infty$, the conditional distribution of M^ψ coincides with that of the supremum of the partial sums of an i.i.d. sequence $(\psi'_n, n \geq 1)$ where

$$P(\psi'_n \in \cdot) := P(\eta_1 - K_0 \in \cdot \mid \delta_1 = 1). \quad (19)$$

It follows from (17) that $E\psi'_1 = -1$. Since $\sum_n \delta_n = \infty$ a.s., it follows that the random variable M^ψ is finite a.s. and does not depend on the joint distribution of the random variables $(\delta_n, \varphi_n, n \geq 1)$. In particular, M^ψ and M^φ are independent random variables. Further, since $F^\dagger \in \mathcal{S} \subset \mathcal{L}$, it follows from (19) that the common distribution F_ψ of the random variables ψ'_n satisfies $\overline{F_\psi^\dagger}(y) \sim \overline{F^\dagger}(y)/\varepsilon$ as $y \rightarrow \infty$. Hence, by the Pakes-Veraverbeke Theorem,

$$P(M^\psi > y) \sim \frac{1}{\varepsilon} \overline{F^\dagger}(y) \quad \text{as } y \rightarrow \infty. \quad (20)$$

We now consider the tail distribution of M^φ and show that this is exponentially bounded. For each n , let F_n^φ be the distribution of φ_n . We show first how to choose a constant s , depending on F, α, β only, such that the process $\exp s S_n^\varphi$ is a supermartingale. For this we require that, for all n ,

$$\frac{1}{s} \int_{-\infty}^{\infty} (e^{sz} - 1) F_n^\varphi(dz) \leq 0. \quad (21)$$

From (13), (14), and our assumption that $\varphi_n \geq -\beta$ a.s., it follows that, for all n ,

$$|\varphi_n| \leq K := \max(\beta, y^*, K_0).$$

From this, and the inequality $e^{sz} \leq 1 + sz + s^2 K^2 e^{sK}$, valid for any $s \geq 0$ and for any z such that $|z| \leq K$, it follows that the left side of (21) is bounded above by $E\varphi_n + sK^2 e^{sK}$, which, by (16), is less than or equal to zero for any $s > 0$ such that $sK^2 e^{sK} \leq -\alpha/4$.

Thus we fix such an s , depending only on F, α, β as required. It now follows by the usual argument involving the martingale maximal inequality that, for $y \geq 0$,

$$P(M^\varphi > y) \leq e^{-sy}. \quad (22)$$

Let ζ be a random variable which has tail distribution e^{-sy} and which is independent of everything else. Since M^φ and M^ψ are independent, it follows from (18) that

$$P(M > y) \leq P(M^\psi + \zeta > y). \quad (23)$$

Further, from Lemma A.1,

$$\lim_{y \rightarrow \infty} \frac{P(M^\psi + \zeta > y)}{\overline{F^I}(y)} = \lim_{y \rightarrow \infty} \frac{P(M^\psi > y)}{\overline{F^I}(y)} = \frac{1}{\varepsilon}.$$

and so there exists r such that, for all $y \geq 0$,

$$P(M^\psi + \zeta > y) \leq r \overline{F^I}(y). \quad (24)$$

Finally, note that the distributions of M^ψ and ζ , and so also the constant r , depend on F , α and β only, so that the required result now follows from (23) and (24). \square

2.3 Proofs of Theorems 2.1 and 2.2

We now return to the model and assumptions of Section 2.1 and prove our main results. We give first the following technical lemma, which will be required subsequently.

Lemma 2.1. *Suppose that $F^I \in \mathcal{L}$ and that d_1, d_2 are constants such that $d_2 > 0$. Then*

$$\sum_{n \geq 1} \overline{F}(y + d_1 + d_2 n) \sim \frac{1}{d_2} \overline{F^I}(y) \quad \text{as } y \rightarrow \infty, \quad (25)$$

and, for any real sequence δ_n such that $\delta_n \downarrow 0$,

$$\sum_{n \geq 1} \delta_n \overline{F}(y + d_1 + d_2 n) = o(\overline{F^I}(y)) \quad \text{as } y \rightarrow \infty. \quad (26)$$

The conditions (D1) and (D2) further imply that

$$\sum_{n \geq 1} \int_{\mathcal{X}} \pi_n(dx) \overline{F}_x(y + d_1 + d_2 n) \sim \frac{C}{d_2} \overline{F^I}(y) \quad \text{as } y \rightarrow \infty. \quad (27)$$

Proof. The results (25) and (26) are elementary consequences of the condition $F^I \in \mathcal{L}$, and, in each case, the approximation of a sum by an integral. Detailed proofs may be found in Foss and Zachary (2002). We prove (27) under the assumption that the regenerative process X

is aperiodic, so that the distance $\|\pi_n - \pi\|$ between π_n and π in the total variation norm tends to zero—the modifications required to deal with the periodic case are routine. Then

$$\sum_{n \geq 1} \int_{\mathcal{X}} \pi_n(dx) \overline{F}_x(y + d_1 + d_2n) \sim \sum_{n \geq 1} \int_{\mathcal{X}} \pi(dx) \overline{F}_x(y + d_1 + d_2n) \quad \text{as } y \rightarrow \infty, \quad (28)$$

since the absolute value of the difference between the left and right sides of (28) is bounded by $\sum_n \|\pi_n - \pi\| \overline{F}(y + d_1 + d_2n)$, which, by (26), is $o(\overline{F^1}(y))$ as $y \rightarrow \infty$. Further, using the condition (D1), for y sufficiently large that $\overline{F^1}(y + d_1) < 1$, the right side of (28) is bounded above and below by

$$\frac{1}{d_2} \int_{\mathcal{X}} \pi(dx) \overline{F}_x^1(y + d_1) \quad \text{and} \quad \frac{1}{d_2} \int_{\mathcal{X}} \pi(dx) \overline{F}_x^1(y + d_1 + d_2)$$

respectively. From the conditions (D1), (D2) and the dominated convergence theorem, for any constant d ,

$$\int_{\mathcal{X}} \pi(dx) \overline{F}_x^1(y + d) \sim \overline{F^1}(y + d) \int_{\mathcal{X}} \pi(dx) c(x) \quad \text{as } y \rightarrow \infty.$$

The result (27) now follows from the condition $F^1 \in \mathcal{L}$ □

The following lemma gives an asymptotic lower bound for $P(M > y)$. This result is also proved in Foss and Zachary (2002). For completeness we give a short, simplified proof in the Appendix to the present paper—see also Zachary (2004).

Lemma 2.2. *Suppose that (D1)–(D3) hold and that $F^1 \in \mathcal{L}$. Then*

$$\liminf_{y \rightarrow \infty} \frac{P(M > y)}{\overline{F^1}(y)} \geq \frac{C}{a}.$$

Remark 2.5. As in the Pakes-Veraverbeke Theorem for unmodulated random walks, the intuitive idea underlying the above result is the following: the maximum M will exceed a large value y if the process follows the typical behaviour specified by the law of large numbers, i.e. it's mean path, except that at any time n a jump occurs of size greater than $y + na$; this has probability $E[\overline{F}_{X_n}(y + na)]$, and so the bound is now given by the use of (27).

We shall argue similarly for the upper bound: if M exceeds a large value y then it *must* be the case that a single jump exceeds y plus the typical behaviour of the process. We now proceed to making this heuristic more precise.

We consider first, in Lemma 2.3 below, the upper bound for the relatively simple case $\kappa < 0$. This result may be combined with the lower bound of Lemma 2.2 to give the exact asymptotics in this case (Theorem 2.1). We then use the *result* of Lemma 2.3 to extend the upper bound, in the proof of Theorem 2.2, to general κ , thereby obtaining the exact asymptotics in this case also.

Lemma 2.3. *Suppose that (D1)–(D3) hold, that $F^1 \in \mathcal{S}$, and that $\kappa < 0$. Then*

$$\limsup_{y \rightarrow \infty} \frac{P(M > y)}{\overline{F^1}(y)} \leq \frac{C}{a}.$$

Proof. For given (small) $\varepsilon > 0$, and (large) $u_0 > 0$, for each $n \geq 0$ define $u_n = u_0 - (a - \varepsilon)n$. Define the stopping time

$$\sigma = \inf\{n \geq 0 : S_n > u_n\}.$$

Since $S_n/n \rightarrow -a$ a.s., it follows that (for fixed ε)

$$P(\sigma < \infty) \rightarrow 0 \quad \text{as } u_0 \rightarrow \infty. \quad (29)$$

Note that S_σ and $M_\sigma = \max_{0 \leq n \leq \sigma} S_n$ are only defined on $\{\sigma < \infty\}$. Here, and elsewhere, we use the convention that any probability of an event involving random variables such as S_σ or M_σ is actually the probability of the same event intersected by $\{\sigma < \infty\}$, e.g. $P(M_\sigma > y) := P(M_\sigma > y, \sigma < \infty)$.

Since $S_n \leq u_n$ for all $n < \sigma$, we have, for $y > u_0$,

$$\{M_\sigma > y\} = \{S_\sigma > y\} = \{S_{\sigma-1} + \xi_\sigma > y\} \subseteq \{\xi_\sigma > y - u_{\sigma-1}\},$$

and hence

$$\begin{aligned} P(M_\sigma > y) &= P(S_\sigma > y) \leq \sum_{n=1}^{\infty} P(\xi_n > y - u_{n-1}) \\ &= \sum_{n=1}^{\infty} \int_{\mathcal{X}} \pi_n(dx) \overline{F}_x(y - u_{n-1}) \\ &\sim \frac{C}{a - \varepsilon} \overline{F}^{\Gamma}(y), \end{aligned} \quad (30)$$

as $y \rightarrow \infty$, where the last equivalence follows from Lemma 2.1.

Since $\kappa < 0$ it follows from (6) that we can choose any $\alpha \in (0, -\kappa)$ and then $\beta > 0$ sufficiently large that

$$\sup_{x \in \mathcal{X}} \int_{\mathbb{R}} (z \vee -\beta) F_x(dz) \leq -\alpha. \quad (31)$$

On the set $\{\sigma < \infty\}$ define the sequence of random variables $(\xi_n^\sigma, n \geq 1)$ by $\xi_n^\sigma = \xi_{\sigma+n}$; let $M^\sigma = \sup_{n \geq 0} \sum_{i=1}^n \xi_i^\sigma$. Conditional on the background process X and any finite value of σ , the sequence $(\xi_n^\sigma, n \geq 1)$ consists of independent random variables which, from (31), satisfy the conditions (9) and (10) of Proposition 2.1 (with F , α and β as defined here). It therefore follows from that proposition that there exists r , depending on F , α and β only, such that, for all x , all finite n and all $y \geq 0$,

$$P(M^\sigma > y \mid X = x, \sigma = n) \leq r \overline{F}^{\Gamma}(y); \quad (32)$$

further, conditional on $X = x$ and $\sigma = n$, the random variables M_σ and M^σ are independent. Let \widetilde{M} be a random variable, *independent of all else*, with tail distribution

$$P(\widetilde{M} > y) = 1 \wedge r \overline{F}^{\Gamma}(y), \quad (33)$$

Observe that, for $y > u_0$, we have $M = S_\sigma + M^\sigma = M_\sigma + M^\sigma$. By conditioning on X and each finite value of σ , it follows from (32), (33) and the above conditional independence that

$$\begin{aligned} P(M > y) &= P(M_\sigma + M^\sigma > y, \sigma < \infty) \\ &\leq P(M_\sigma + \widetilde{M} > y, \sigma < \infty) \\ &= P(\sigma < \infty) P(M_\sigma + \widetilde{M} > y \mid \sigma < \infty) \end{aligned} \quad (34)$$

Also, from (30),

$$\overline{\lim}_{y \rightarrow \infty} \frac{P(M_\sigma > y | \sigma < \infty)}{F^{\mathbb{I}}(y)} \leq \frac{C}{(a - \varepsilon)P(\sigma < \infty)}. \quad (35)$$

From (33), (34), (35), the independence of \widetilde{M} from all else, and Lemma A.1,

$$\begin{aligned} \overline{\lim}_{y \rightarrow \infty} \frac{P(M > y)}{F^{\mathbb{I}}(y)} &\leq P(\sigma < \infty) \left(\frac{C}{(a - \varepsilon)P(\sigma < \infty)} + r \right) \\ &= \frac{C}{a - \varepsilon} + rP(\sigma < \infty). \end{aligned} \quad (36)$$

It follows from (29) that as $u_0 \rightarrow \infty$ the second term on the right side of (36) tends to 0. The required result now follows on letting also $\varepsilon \rightarrow 0$. \square

Proof of Theorem 2.1. This is now immediate from Lemmas 2.2 and 2.3. \square

Proof of Theorem 2.2. Let nonnegative $b > \kappa$ be such that the condition (D4) holds. Choose

$$\delta \in (0, \min(a, b - \kappa)) \quad (37)$$

and choose $\varepsilon \in (0, a - \delta)$. Note that, from the condition (D3),

$$\int_{\mathcal{X}} (a_x + \delta)\pi(dx) = -a + \delta < -\varepsilon.$$

It now follows from the definition (6) of κ , and since also $b > \kappa$, that we may choose $\beta > 0$ sufficiently large that

$$\int_{\mathcal{X}} (a_x^\beta + \delta)\pi(dx) < -\varepsilon, \quad (38)$$

$$a_x^\beta + \delta \leq b \quad \text{for all } x \in \mathcal{X}, \quad (39)$$

(where a_x^β is as defined by (5)). Let b_x be a measurable function on \mathcal{X} such that, for some sufficiently large $d > 0$,

$$\max(-d, a_x^\beta + \delta) \leq b_x \leq b, \quad x \in \mathcal{X}, \quad (40)$$

$$\int_{\mathcal{X}} b_x \pi(dx) = -\varepsilon. \quad (41)$$

(To see that such a function b_x exists, note that, from (38), we may choose d sufficiently large that

$$\int_{\mathcal{X}} \max(-d, a_x^\beta + \delta)\pi(dx) < -\varepsilon;$$

since also $b > -\varepsilon$, we may, for example, satisfy (40) and (41) by choosing $b_x = \max(s, -d, a_x^\beta + \delta)$ for the appropriate constant $s \in (-d, b)$.)

Define, for each $n \geq 1$,

$$\hat{\xi}_n = \xi_n - b_{X_n}. \quad (42)$$

Note that, conditional on the modulating process X , the random variable $\hat{\xi}_n$ has distribution function \hat{F}_{X_n} where, for each $x \in \mathcal{X}$ and each $y \in \mathbb{R}$, $\hat{F}_x(y) = F_x(y + b_x)$. Since also $F^{\mathbb{I}} \in \mathcal{S} \subset \mathcal{L}$, the family of distributions $(\hat{F}_x, x \in \mathcal{X})$ satisfies the conditions (D1) and (D2) with F replaced by \hat{F} where $\hat{F}(y) = F(y - d)$.

Since also, for each $x \in \mathcal{X}$, the distribution \hat{F}_x has mean $\hat{a}_x := a_x - b_x$ and $b_x \leq b$, it follows that, for $\beta' = b + \beta$,

$$\int_{\mathbb{R}} (z \vee -\beta') \hat{F}_x(dz) \leq a_x^\beta - b_x \leq -\delta, \quad (43)$$

where the second inequality above also follows from (40). Lastly, it follows from the condition (D3) and (41) that, for each $x \in \mathcal{X}$,

$$\int_{\mathcal{X}} \hat{a}_x \pi(dx) = -a + \varepsilon < 0.$$

The process $(\hat{S}_n, n \geq 0)$ given by $\hat{S}_n = \sum_{i=1}^n \hat{\xi}_i$, for each $n \geq 0$, thus satisfies all the conditions associated with Lemma 2.3, where F is replaced by \hat{F} , κ is replaced by the appropriate $\hat{\kappa}$ with, from (43), $\hat{\kappa} \leq -\delta$, and a is replaced by $a - \varepsilon$. Since also the condition $F^\Gamma \in \mathcal{S}$ implies that $\hat{F}^\Gamma \in \mathcal{S} \subset \mathcal{L}$ (and that in particular \hat{F}^Γ is tail-equivalent to F^Γ), we conclude that the supremum \hat{M} of the process $(\hat{S}_n, n \geq 0)$ satisfies

$$\overline{\lim}_{y \rightarrow \infty} \frac{P(\hat{M} > y)}{\overline{F}^\Gamma(y)} \leq \frac{C}{a - \varepsilon}. \quad (44)$$

It also follows from (43) that the family of distributions $(\hat{F}_x, x \in \mathcal{X})$ satisfies the conditions (9) and (10) of Proposition 2.1 with F replaced by \hat{F} , α replaced by δ , and β by $\beta + b$. Hence, again since $\hat{F}^\Gamma \in \mathcal{S}$ and is tail-equivalent to F^Γ , there exists a constant r such that, for all $x \in \mathcal{X}$, and for all y ,

$$P(\hat{M} > y \mid X = x) \leq \min(1, r \overline{F}^\Gamma(y)). \quad (45)$$

Define also the process $(S_n^b, n \geq 0)$ by $S_n^b = \sum_{i=1}^n b_{X_i}$ for $n \geq 0$. Let $\eta_0 = S_{T_0}^b$ and $\eta_k = S_{T_k}^b - S_{T_{k-1}}^b$, $k \geq 1$, be the increments of this process between the successive regeneration epochs of the modulating process X . It follows from (40) that, for each $k \geq 0$, $\eta_k \leq b\tau_k$. For a constant K to be specified below, define $\zeta_k = \max(\eta_k, b\tau_k - K)$ for each $k \geq 0$. The random variables ζ_k are independent for $k \geq 0$, and are identically distributed for $k \geq 1$; let K be such that

$$E\zeta_k < 0, \quad k \geq 1. \quad (46)$$

Note also that $\zeta_k \leq b\tau_k$ for each $k \geq 0$ and so, from the condition (D4),

$$P(\zeta_0 > y) = o(\overline{F}^\Gamma(y)), \quad P(\zeta_1 > y) = o(\overline{F}^\Gamma(y)), \quad \text{as } y \rightarrow \infty. \quad (47)$$

Now let $M^b = \sup_{n \geq 0} S_n^b$. Then

$$\begin{aligned} M^b &\leq \sup(b\tau_0, \eta_0 + b\tau_1, \eta_0 + \eta_1 + b\tau_2, \dots) \\ &\leq K + \sup(\zeta_0, \zeta_0 + \zeta_1, \zeta_0 + \zeta_1 + \zeta_2, \dots) \\ &\leq K + \zeta_0 + \sup(0, \zeta_1, \zeta_1 + \zeta_2, \dots). \end{aligned} \quad (48)$$

It follows from (46), (47), the independence of the random variables ζ_k , $k \geq 0$, and the Pakes-Veraverbeke Theorem that the last term on the right side of (48) has a probability of exceeding y which is $o(\overline{F}^\Gamma(y))$ as $y \rightarrow \infty$. It now follows, from (47), (48), the above independence and Lemma A.1, that

$$\lim_{y \rightarrow \infty} \frac{P(M^b > y)}{\overline{F}^\Gamma(y)} = 0. \quad (49)$$

Finally, note that, for each n , we have $S_n = \hat{S}_n + S_n^b$ and hence $M \leq \hat{M} + M^b$. Since \hat{M} and M^b are conditionally independent given X , it follows from (44), (45), (49) and Lemma A.2 that

$$\overline{\lim}_{y \rightarrow \infty} \frac{P(M > y)}{\overline{F^1}(y)} \leq \overline{\lim}_{y \rightarrow \infty} \frac{P(\hat{M} + M^b > y)}{\overline{F^1}(y)} \leq \frac{C}{a - \varepsilon}. \quad (50)$$

By letting $\varepsilon \rightarrow 0$ in (50) and combining this result with the lower bound given by Lemma 2.2, we now obtain the required result. \square

2.4 Example

We give here an example to show the necessity of the condition (D4).

Example 2.1. Let $\zeta, \zeta_i, i \geq 1$, be i.i.d. non-negative random variables with common distribution function F . Assume that $E\zeta = 1$ and that $F^1 \in \mathcal{S}$.

We take the modulating process $X = (X_n, n \geq 1)$ to be an independent Markov chain on $\mathbb{Z}_+ = \{0, 1, \dots\}$ with initial value $X_1 = 0$ and transition probabilities $p_{0,0} = 0, p_{0,j} > 0$ for all $j \geq 1$ and, for $j \geq 1, p_{j,j-1} = 1$. Define $T_0 = 0$ and for $k \geq 1, T_k = \min\{n > T_{k-1} : X_n = 1\}$. We regard $T_k, k \geq 0$, as the regeneration times of the process. Since $p_{1,0} = 1$, it follows that, for $k \geq 1$, the k th cycle starts at time $T_{k-1} + 1$ in state 0, and further that the cycle lengths $\tau_k = T_k - T_{k-1}$, are i.i.d. random variables with a distribution concentrated on $\{2, 3, \dots\}$ and distribution function G given by $G(y) = \sum_{j \leq y-1} p_{0,j}$. Assume further that $E\tau_1 = 1 + \sum_{j \geq 1} j p_{0,j} < \infty$. Then the Markov chain X is ergodic.

Now define the modulated random walk $(S_n, n \geq 0)$ by $S_0 = 0$ and $S_n = \sum_{i=1}^n \xi_i$ where the random variables ξ_i are given by

$$\xi_i = \zeta_i - d \mathbf{1}(X_i = 0)$$

for some constant $d > E\tau_1$. The conditions (D1)–(D3) are thus satisfied with F as defined here, $c(x) = 1$ for all x , and $a = d/E\tau - 1$.

Since the random variables ζ_i are nonnegative, we have $S_n \geq S_{n-1}$ for all n such that $X_n \neq 0$, i.e. for all $n \neq T_{k-1} + 1$ for some k . It follows that

$$M := \sup_{n \geq 0} S_n = \sup_{m \geq 0} \left(\sum_{k=1}^m \psi_k \right) \quad (51)$$

where, for $k \geq 1$,

$$\psi_k := \sum_{i=T_{k-1}+1}^{T_k} \zeta_i - d$$

are i.i.d. random variables with common negative mean $E\tau - d$.

For the process $(S_n, n \geq 0)$ here, the constant κ defined by (6) is given $\kappa = E\zeta = 1$. For an arbitrary $b < 1$, we provide an example when $P(b\tau > y) = o(\overline{F}(y))$, but for which $\overline{F^1}(y) = o(P(M > y))$, in each case as $y \rightarrow \infty$. Thus in this case the conclusion of Theorem 2.2 cannot hold.

Choose $\gamma \in (0, 1)$ and suppose that $\overline{F}(y) = e^{-y^\gamma}$ (for which it is well-known that $F^1 \in \mathcal{S}$). Suppose also that $P(\tau > y) \sim e^{-cy^\gamma}$ as $y \rightarrow \infty$ for some $c \in (b^\gamma, 1)$. Then it is readily checked that $P(b\tau > y) = o(\overline{F}(y))$. For any fixed $\varepsilon \in (0, 1)$ such that $(1 - \varepsilon)^\gamma > c$, define the distribution H by

$$\overline{H}(y) = \exp\left(-\frac{cy^\gamma}{(1 - \varepsilon)^\gamma}\right).$$

We now have

$$\begin{aligned} P(\psi_1 > y) &\geq P\left(\left|\frac{\sum_1^n \zeta_i}{n} - 1\right| \leq \varepsilon \quad \forall n > y + d\right) P\left(\tau_1 > \frac{y + d}{1 - \varepsilon}\right) \\ &\sim P\left(\tau_1 > \frac{y + d}{1 - \varepsilon}\right) \end{aligned} \tag{52}$$

$$\sim \bar{H}(y), \tag{53}$$

as $y \rightarrow \infty$, where (52) follows by the Strong Law of Large numbers, and (53) follows since $H \in \mathcal{L}$. Since also $H^\dagger \in \mathcal{S}$, it follows from (51), (53), and the Pakes-Veraverbeke Theorem (by for example noting that each random variable ψ_k stochastically dominates a random variable ψ'_k such that $P(\psi'_1 > y) \sim \bar{H}(y)$) that

$$\liminf_{y \rightarrow \infty} \frac{P(M > y)}{\bar{H}^\dagger(y)} \geq 1.$$

Finally, since also $\bar{F}(y) = o(\bar{H}(y))$, and so also $\bar{F}^\dagger(x) = o(\bar{H}^\dagger(y))$, as $y \rightarrow \infty$, it follows that $\bar{F}^\dagger(x) = o(P(M > y))$ as required.

Finally, we remark that while this example may be simplified somewhat by assuming the random variables ζ_i to be a.s. constant, we have some hope that, for a suitable choice of F , we may show the necessity of the *strict* inequality $b > \kappa$ in the condition (D4).

3 Modulated random walk in continuous time

In this section we consider a continuous-time process $(S_t, t \geq 0)$, whose increments are independent and modulated by a background process $X = (X_t, t \geq 0)$ with a regenerative structure. Analogously to the discrete-time theory, the process is assumed to have jumps which are heavy tailed and that $S_t \rightarrow -\infty$ as $t \rightarrow \infty$. We are again interested in the asymptotic form of the tail distribution of the maximum of the process. Many of the probabilistic ideas are similar to the ones before. However, we need to define the processes carefully and we do so in Section 3.1. We then present the main results in Section 3.2, a general result for processes with independent (but non-stationary) increments in Section 3.3, followed by the proofs in Section 3.4. We refer to Kallenberg (2002, Ch. 15) for the theory and construction of processes with independent increments.

3.1 Definitions

A process with independent increments. We define what we mean by a process $(S_t, t \geq 0)$ with $S_0 = 0$, independent increments and distribution specified by a triple

$$(\nu(t, \cdot), v(t)^2, a(t), \quad t \geq 0).$$

First, $t \mapsto a(t)$ (respectively $t \mapsto v(t)^2$) is a real-valued (respectively positive) function that is integrable over finite intervals. Second, for each t , the quantity $\nu(t, \cdot)$ is a Borel measure on \mathbb{R} with $\nu(t, \{0\}) = 0$ and $\int_{\mathbb{R}} (y^2 \wedge |y|) \nu(t, dy) < \infty$; also, for each Borel set B , the function $t \mapsto \nu(t, B)$ is integrable over finite intervals.

Next let Φ be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $E\Phi(dt, dy) = dt \nu(t, dy)$. Note that the intensity measure is sigma-finite and so the Poisson random measure is well-defined.

Finally, for each t , let $A_t := \int_0^t a(s)ds$; let $(W_t, t \geq 0)$ be a zero-mean Gaussian process with independent increments and $\text{var } W_t = \int_0^t v(s)^2 ds$, and, for each $t \geq 0$, let $Y_t = \int_{[0,t] \times \mathbb{R}} y[\Phi(ds, dy) - ds \nu(s, dy)]$. Note that the process $(Y_t, t \geq 0)$ is centred so that $EY_t = 0$ for all t . Set

$$S_t = A_t + W_t + Y_t, \quad t \geq 0.$$

Thus, $S = (S_t, t \geq 0)$ is a process with independent increments (see, e.g., Kallenberg (2002)) and, in particular $ES_t = A_t$ for all t . It is not the most general version of a process with independent increments, because we assumed that (i) its mean $ES_t = A_t$ exists (ii) the functions $t \mapsto ES_t$, $t \mapsto EW_t^2$ are absolutely continuous, and (iii) the intensity measure $E\Phi(dt, dy)$ has density with respect to the first coordinate. (Note that while the assumptions (ii) and (iii) are essentially technical, the assumption (i) is essential; in its absence we would need to pursue a different treatment—in the spirit of Klüppelberg, Kyprianou and Maller (2004) and of Denisov, Foss and Korshunov (2004).)

A modulated continuous-time random walk. Next assume that we are given a regenerative process $X = (X_t, t \geq 0)$ such that X_t takes values in some measurable space $(\mathcal{X}, \mathcal{X}^c)$, a measurable real-valued function $(a_x, x \in \mathcal{X})$, a measurable positive function $(v_x^2, x \in \mathcal{X})$, and a collection of measures $(\nu_x(\cdot), x \in \mathcal{X})$, such that $x \mapsto \nu_x(B)$ is measurable for each Borel set $B \subseteq \mathbb{R}$.

For each sample path $(X_t, t \geq 0)$, define $S = (S_t, t \geq 0)$ as being a process generated by the triple

$$(\nu(t, \cdot), v(t)^2, a(t)) := (\nu_{X_t}(\cdot), v_{X_t}^2, a_{X_t}).$$

As above we assume that, for each t , and each Borel set B ,

$$\int_0^t a_{X_s} ds, \quad \int_0^t v_{X_s}^2 ds, \quad \int_0^t \nu_{X_s}(B) ds \quad \text{are a.s. finite,} \quad (54)$$

and that, for each $x \in \mathcal{X}$,

$$\nu_x(\{0\}) = 0, \quad \int_{\mathbb{R}} (y^2 \wedge |y|) \nu_x(dy) < \infty.$$

(For example we note that, if \mathcal{X} is generated by some topology, a sufficient condition for (54) to hold is that X have càdlàg paths and that $x \mapsto a_x$, etc. be continuous functions.)

We can construct this process by considering a family $(\Phi_x, x \in \mathcal{X})$ of Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure

$$E\Phi_x(dt, dy) = dt \nu_x(dy),$$

and an independent standard Brownian motion $(B_t, t \geq 0)$. We set

$$\begin{aligned} A_t &:= \int_0^t a_{X_s} ds \\ W_t &:= \int_0^t v_{X_s} dB_s \\ Y_t &:= \int_0^t \int_{\mathbb{R}} y[\Phi_{X_s}(ds, dy) - \nu_{X_s}(dy)ds] \\ S_t &:= A_t + W_t + Y_t. \end{aligned}$$

We then have that, for each t , the characteristic function of S_t , conditional on the background process X , is

$$E[e^{i\theta S_t} | X] = \exp \left\{ i\theta \int_0^t a_{X_s} ds - \frac{\theta^2}{2} \int_0^t v_{X_s}^2 ds + \int_0^t ds \int_{\mathbb{R}} \nu_{X_s}(dy) [e^{i\theta y} - 1 - i\theta y] \right\}.$$

We shall refer to $S = (S_t, t \geq 0)$ as a modulated continuous-time random walk. We assume that we choose a version of S with càdlàg paths. (The reader will recognise that, in absence of modulation, the last formula is the Lévy-Khinchine formula for a Lévy process—see Bertoin (1998) or Sato (2000).) We shall use the notation ΔS_t for the size of the jump at any time t , i.e.

$$\Delta S_t := S_t - S_{t-}.$$

We will also need to denote by Φ the point process on $\mathbb{R}_+ \times \mathbb{R}$ with atoms the pairs $(t, \Delta S_t)$, for those t for which $\Delta S_t \neq 0$, i.e. $\Phi(B) := \sum_{t: \Delta S_t \neq 0} \mathbf{1}((t, \Delta S_t) \in B)$, $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$. Then, conditional on X , Φ is a Poisson point process with intensity measure

$$\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \ni B \mapsto E[\Phi(B) | X] = \iint_B dt \nu_{X_t}(dy). \quad (55)$$

3.2 Main results

We assume that the process $(S_t, t \geq 0)$ is modulated by a regenerative background process $X = (X_t, t \geq 0)$ as described in the previous section.

Denote the regeneration epochs of X by $0 \leq T_0 < T_1 < \dots$. By definition, the cycles $((X_t, T_{k-1} < t \leq T_k), k \geq 1)$ are i.i.d. and independent of the initial cycle $(X_t, 0 < t \leq T_0)$. Define $\tau_0 = T_0$, $\tau_k = T_k - T_{k-1}$, $k \geq 1$. Then $(\tau_k, k \geq 0)$ are independent and $(\tau_k, k \geq 1)$ are identically distributed. We assume that $E\tau_1 < \infty$. For each $t \geq 0$, let π_t be the distribution of X_t , and let π denote the stationary probability measure

$$\pi(B) := \frac{1}{E\tau_1} E \int_{T_1}^{T_2} \mathbf{1}(X_t \in B) dt, \quad B \in \mathcal{B}(\mathcal{X}).$$

We require the extension of some definitions from distributions to measures. For any positive measure ν on \mathbb{R} , again satisfying the conditions

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}} (y^2 \wedge |y|) \nu(dy) < \infty, \quad (56)$$

we write $\bar{\nu}(y) := \nu((y, \infty))$ for all $y > 0$. We say that ν is *subexponential*, and write $\nu \in \mathcal{S}$, if and only if $\bar{\nu}(y) \sim c\bar{F}(y)$ as $y \rightarrow \infty$ for some distribution $F \in \mathcal{S}$ and constant $c > 0$, i.e. if and only if ν has a subexponential tail; we similarly say that ν is *long-tailed*, and write $\nu \in \mathcal{L}$, if and only if $\bar{\nu}(y) \sim c\bar{F}(y)$ as $y \rightarrow \infty$ for some distribution $F \in \mathcal{L}$ and constant $c > 0$. Hence here also we have $\mathcal{S} \subset \mathcal{L}$. Finally, we define the *integrated* (or *second*) *tail* measure ν^{\dagger} on $\mathbb{R}_+ \setminus \{0\}$ by $\bar{\nu}^{\dagger}(y) = \int_y^{\infty} \bar{\nu}(z) dz < \infty$ for all $y > 0$.

The family $(\nu_x, v_x^2, a_x, x \in \mathcal{X})$ specifying the distribution of $(S_t, t \geq 0)$ will be assumed to satisfy the following additional conditions with respect to some *reference* measure ν on \mathbb{R}

satisfying (56), some measurable function $c : \mathcal{X} \rightarrow [0, 1]$ and constants γ and v^2 :

- (C1) $\bar{\nu}_x(y) \leq \bar{\nu}(y)$ for all $y > 0$, $x \in \mathcal{X}$,
- (C2) $\bar{\nu}_x^1(y) \sim c(x)\bar{\nu}^1(y)$ as $y \rightarrow \infty$, $x \in \mathcal{X}$,
- (C3) $\int_{-\infty}^{\infty} (1 \wedge y^2) \nu_x(dy) \leq \gamma$, $x \in \mathcal{X}$,
- (C4) $v_x^2 \leq v^2$, $x \in \mathcal{X}$,
- (C5) $a := -\int_{\mathcal{X}} a_x \pi(dx)$ is finite and strictly positive.

Remark 3.1. The conditions (C1), (C2) and (C5) are analogous to those of the discrete-time conditions (D1), (D2) and (D3). The remaining conditions (C3) and (C4) are additional, and very natural, uniformity conditions necessitated by the continuous-time environment and have no (nontrivial) discrete-time analogues. (With regard to the condition (C3), note that the uniform boundedness in x of $\bar{\nu}_x(1)$ is already guaranteed by the condition (C1); the formulation of (C3) as above is for convenience.) Remarks 2.2–2.4 in Section 2 have obvious counterparts here. We further remark that the condition imposed by (C3) on the left tails of the measures ν_x may be weakened at the expense of some additional technical complexity. Finally, note that only the restriction of the measure ν to $\mathbb{R}_+ \setminus \{0\}$ is relevant to the above conditions.

As in Section 2, it follows from the above conditions that the process $(S_t, t \geq 0)$ then satisfies

$$\frac{S_t}{t} \rightarrow -a, \quad \text{as } t \rightarrow \infty, \quad \text{a.s.} \quad (57)$$

(see the discussion of this result in the Appendix). Hence also $S_t \rightarrow -\infty$, as $t \rightarrow \infty$, a.s., and so $M := \sup_{t \geq 0} S_t$ is finite a.s.

For each $x \in \mathcal{X}$ and $\beta > 0$, define

$$a_x^\beta := a_x - \int_{-\infty}^{-\beta} (y + \beta) \nu_x(dy). \quad (58)$$

(Here $\int_{-\infty}^{-\beta}$ denotes $\int_{(-\infty, -\beta]}$; we use similar conventions elsewhere.) Define also

$$\kappa := \lim_{\beta \rightarrow \infty} \sup_{x \in \mathcal{X}} a_x^\beta. \quad (59)$$

As in discrete time, in the case where the measures ν_x , $x \in \mathcal{X}$, satisfy the uniform integrability condition

$$\lim_{\beta \rightarrow \infty} \sup_{x \in \mathcal{X}} \int_{-\infty}^{-\beta} |y| \nu_x(dy) = 0,$$

it follows from (58) and (59) that $\kappa = \sup_{x \in \mathcal{X}} a_x$. Define $C \in [0, 1]$ by

$$C := \int_{\mathcal{X}} c(x) \pi(dx). \quad (60)$$

Theorem 3.1 below, for the case $\kappa < 0$, is the analogue of Theorem 2.1 for the discrete-time case.

Theorem 3.1. *Suppose that (C1)–(C5) hold, that $\nu^1 \in \mathcal{S}$, and that $\kappa < 0$. Then*

$$\lim_{y \rightarrow \infty} \frac{P(M > y)}{\bar{\nu}^1(y)} = \frac{C}{a}.$$

For the case where the sign of κ may be arbitrary, we again require an additional condition, similar to (D4), regarding the (tail) distributions of the lengths of the regenerative cycles. The condition here is:

(C6) For some nonnegative $b > \kappa$,

$$P(b\tau_0 > t) = o(\bar{\nu}^1(t)), \quad P(b\tau_1 > t) = o(\bar{\nu}(t)), \quad \text{as } t \rightarrow \infty. \quad (61)$$

As in the discrete-time case, for $\kappa < 0$ the condition (C6) is trivially satisfied by taking $b = 0$, so that again Theorem 3.1 may be viewed as a special case of the general result given by Theorem 3.2 below.

Theorem 3.2. *Suppose that (C1)–(C6) hold and that $\nu^1 \in \mathcal{S}$. Then*

$$\lim_{y \rightarrow \infty} \frac{P(M > y)}{\bar{\nu}^1(y)} = \frac{C}{a}.$$

3.3 A uniform upper bound for continuous-time processes with independent increments

We prove in this section an auxiliary proposition, analogous to that of Proposition 2.1 for the discrete-time case, which will be required for the upper bound.

Proposition 3.1. *Let ν be a Borel measure on \mathbb{R} satisfying (56) and such that $\nu^1 \in \mathcal{S}$. For strictly positive constants $\alpha, \beta, \gamma, v^2$, let the process $(S_t, t \geq 0)$ have distribution given by a triple $(\nu_t, v_t^2, a_t, t \geq 0)$ satisfying the conditions of Section 3.1 and such that, for all t ,*

$$\bar{\nu}_t(y) \leq \bar{\nu}(y) \quad \text{for all } y > 0, \quad (62)$$

$$\int_{-\infty}^{\infty} (1 \wedge y^2) \nu_t(dy) \leq \gamma, \quad (63)$$

$$v_t^2 \leq v^2, \quad (64)$$

$$a_t - \int_{-\infty}^{-\beta} (y + \beta) \nu_t(dy) \leq -\alpha. \quad (65)$$

Let $M := \sup_{t \geq 0} S_t$. Then there exists a constant r depending only on $\nu, \alpha, \beta, \gamma$ and v^2 such that

$$P(M > y) \leq r\bar{\nu}^1(y) \quad \text{for all } y \geq 0. \quad (66)$$

Proof. Consider any process $(S_t, t \geq 0)$ with distribution given by $(\nu_t, v_t^2, a_t, t \geq 0)$ as above. Choose $\varepsilon \in (0, \alpha/2)$ and $y^* > 0$ sufficiently large that

$$\bar{\nu}(y^*) \leq \varepsilon. \quad (67)$$

Define, for each t , the measure ν_t^u by $\bar{\nu}_t^u(y) := \bar{\nu}_t(y^* \vee y)$ —so that ν_t^u is the restriction of the measure ν_t to (y^*, ∞) ; define also, for each t , the (positive) measure ν_t^l by $\nu_t^l := \nu_t - \nu_t^u$ —so that ν_t^l is the restriction of the measure ν_t to $(-\infty, y^*]$.

Decompose the process $(S_t, t \geq 0)$ as $S_t = S_t^u + S_t^l$, where $S_0^u = S_0^l = 0$, the process $(S_t^u, t \geq 0)$ has distribution given by $(\nu_t^u, 0, -2\varepsilon, t \geq 0)$, and the process $(S_t^l, t \geq 0)$ is independent of $(S_t^u, t \geq 0)$ and has distribution given by $(\nu_t^l, v_t^2, a_t + 2\varepsilon, t \geq 0)$. Define also $M^u := \sup_{t \geq 0} S_t^u$ and $M^l := \sup_{t \geq 0} S_t^l$. Then M^u and M^l are independent and

$$M \leq M^u + M^l. \quad (68)$$

We now obtain upper bounds on the tail distributions of M^u and M^l which, in each case, depend only on ν , α , β , γ and v^2 .

Define the measure ν^* concentrated on (y^*, ∞) by $\bar{\nu}^*(y) := \bar{\nu}(y^* \vee y)$ for each $y > 0$. Since, for each t , ν_t^u is the restriction of the measure ν_t to (y^*, ∞) and similarly ν^* is the restriction of the measure ν to (y^*, ∞) , it follows from (62) and (67) that, for all $y > 0$, we have $\bar{\nu}_t^u(y) \leq \bar{\nu}^*(y) \leq \varepsilon$. Since also $ES_t^u = -2\varepsilon t$ for all t , it follows (see Section 3.1) that we may couple the process $(S_t^u, t \geq 0)$ with a process $(S_t^*, t \geq 0)$, with $S_0^* = 0$ and distribution given by the time-homogeneous triple $(\nu^*, 0, -\varepsilon)$, in such a way that, almost surely,

$$S_t^u \leq S_t^* \quad \text{for all } t. \quad (69)$$

Define $M^* := \sup_{t \geq 0} S_t^*$. The process $(S_t^*, t \geq 0)$ has i.i.d. positive jumps occurring as a Poisson process with rate $\bar{\nu}^*(y^*) = \bar{\nu}(y^*) \leq \varepsilon$, and is linearly decreasing between these jumps (i.e. it is a compound Poisson process with the subtraction of a linear function). Let the random variables $0 = t_0 < t_1 < t_2 < \dots$ denote the successive jump times. Then the increments $\xi_n^* = S_{t_n}^* - S_{t_{n-1}}^*$, $n \geq 1$, of the process at the successive jump times are i.i.d. random variables. Since also, $ES_t^* = -\varepsilon t$ for all t , we have $\bar{\nu}(y^*)E\xi_1^* \leq -\varepsilon$ and so $E\xi_1^* \leq -1$. Further the jumps of the process $(S_t^*, t \geq 0)$ have a distribution G such that $\bar{G}(y) = \bar{\nu}(y^* \vee y)/\bar{\nu}(y^*)$. Since, as observed, the process is strictly decreasing between these jumps and since $\nu^1 \in \mathcal{S} \subset \mathcal{L}$, it now follows from Lemma A.3 that the distribution H of ξ_1^* is such that $\bar{H}^1(y) \sim \bar{G}^1(y)$ as $y \rightarrow \infty$. Hence, by the Pakes-Veraverbeke Theorem, there exists $r^* > 0$ such that, for all $y \geq 0$,

$$P(M^* > y) \leq r^* \bar{\nu}^1(y). \quad (70)$$

We now consider the tail distribution of M^l , and show that this is exponentially bounded. We show how to choose $s > 0$, depending only on ν , α , β , γ and v^2 , such that the process $(e^{sS_t^l}, t \geq 0)$ is a supermartingale. For this we require (from the distribution of $(S_t^l, t \geq 0)$) that, for all t ,

$$\frac{1}{s} \int_{-\infty}^{\infty} (e^{sy} - 1 - sy) \nu_t^l(dy) + a_t + 2\varepsilon + \frac{v_t^2}{2} s \leq 0. \quad (71)$$

Define $K := \max(y^*, \beta, 1)$. We now use the upper bound, valid for any $s > 0$,

$$\frac{1}{s}(e^{sy} - 1 - sy) \leq \begin{cases} \frac{1}{s}(e^{-s\beta} - 1 - sy) \leq -y - \beta + \frac{\beta^2}{2}s, & y \leq -\beta, \\ se^{sK}y^2, & -\beta < y \leq y^*. \end{cases}$$

Since also ν_t^l is the restriction of the measure ν_t to $(-\infty, y^*]$, it follows that the left side of (71) is bounded above by

$$\begin{aligned} \int_{-\infty}^{-\beta} \left(-y - \beta + \frac{\beta^2}{2}s\right) \nu_t(dy) + se^{sK} \int_{-\beta}^{y^*} y^2 \nu_t(dy) + a_t + 2\varepsilon + \frac{v_t^2}{2} s \\ \leq -\alpha + 2\varepsilon + s \left(\frac{\beta^2}{2} \gamma + e^{sK} K^2 \gamma + \frac{v^2}{2} \right), \end{aligned} \quad (72)$$

by (63)–(65), since, in particular, $y^2 \leq K^2(1 \wedge y^2)$ on the interval $(-\beta, y^*]$. Finally, since $2\varepsilon < \alpha$, it follows that s may be chosen sufficiently small (and dependent only on ν , α , β , γ and v^2) that the right side of (72) is negative.

As in the proof of Proposition 2.1, it now follows, by the usual argument involving the martingale maximal inequality, that, for s as above and $y \geq 0$,

$$P(M^l > y) \leq e^{-sy}. \quad (73)$$

Now let ζ be random variable, independent of all else, which has tail distribution e^{-sy} . From (68), and since M^u and M^l are independent and (by construction) $M^u \leq M^*$ a.s., it follows that, for $y \geq 0$,

$$P(M > y) \leq P(M^u + M^l > y) \leq P(M^u + \zeta > y) \leq P(M^* + \zeta > y). \quad (74)$$

Again, as in the proof of Proposition 2.1, it follows from the independence of M^* and ζ , (70) and (73), and Lemma A.1 that there exists r , depending only on ν , α , β , γ and v^2 , such that, for all $y > 0$,

$$P(M^* + \zeta > y) \leq r\bar{\nu}^1(y),$$

and the required result now follows on using (74). \square

3.4 Proofs of Theorems 3.1 and 3.2

The following Lemma is analogous to Lemma 2.1. Its proof is entirely similar and so will be omitted.

Lemma 3.1. *Suppose that $\nu^1 \in \mathcal{L}$ and that d_1, d_2 are constants such that $d_2 > 0$. Then*

$$\int_0^\infty dt \bar{\nu}(y + d_1 + d_2 t) \sim \frac{1}{d_2} \bar{\nu}^1(y) \quad \text{as } y \rightarrow \infty. \quad (75)$$

The conditions (C1) and (C2) further imply that

$$\int_0^\infty dt \int_{\mathcal{X}} \pi_t(dx) \bar{\nu}_x(y + d_1 + d_2 t) \sim \frac{C}{d_2} \bar{\nu}^1(y) \quad \text{as } y \rightarrow \infty. \quad (76)$$

The following lemma gives an asymptotic lower bound for $P(M > y)$.

Lemma 3.2. *Suppose that (C1)–(C5) hold and that $\nu^1 \in \mathcal{L}$. Then*

$$\liminf_{y \rightarrow \infty} \frac{P(M > y)}{\bar{\nu}^1(y)} \geq \frac{C}{a}.$$

Proof. The proof of this is similar to that of Lemma 2.2. Given $\varepsilon > 0$, by the weak law of large numbers we may choose a constant l_0 sufficiently large that if, for each t , we define $l_t = l_0 + (a + \varepsilon)t$, then

$$P(S_t > -l_t) > 1 - \varepsilon. \quad (77)$$

Recall that, for each $t \geq 0$, $S_{t-} := \lim_{u \uparrow t} S_u$ and $\Delta S_t := S_t - S_{t-}$; define also $M_{t-} := \sup_{u < t} S_u$. For each fixed $y \geq 0$, note that the events

$$A_t := \{M_{t-} \leq y, S_{t-} > -l_t, \Delta S_t > y + l_t\}$$

(defined for all $t \geq 0$) are disjoint—since each $A_t \subseteq \{M_{t-} \leq y, M_t > y\}$. Also, for each t , we have $A_t \subseteq \{M > y\}$. Further, conditional on the background process X , for each t , the events $\{M_{t-} \leq y, S_{t-} > -l_t\}$ and $\{\Delta S_t > y + l_t\}$ are independent. It follows that, for $y \geq 0$,

$$P(M > y) \geq P\left(\bigcup_{t \geq 0} A_t\right) = \int_0^\infty E[\mathbf{1}(\{M_{t-} \leq y, S_{t-} > -l_t\}) \bar{\nu}_{X_t}(y + l_t)] dt. \quad (78)$$

(To obtain this result, we condition on the first (and only) time t such that $\mathbf{1}_{A_t} = 1$, and also use the fact that, conditional on X , the intensity measure of the point process Φ introduced in Section 3.1 is as given by (55).)

Now use the inequality, $E\mathbf{1}_A Z \geq EZ - cP(A^c)$, true for a random variable Z such that $|Z| \leq c$, a.s., to estimate the integrand in (78) as

$$\begin{aligned} E[\mathbf{1}(\{M_{t-} \leq y, S_{t-} > -l_t\})\overline{\nu}_{X_t}(y+l_t)] \\ \geq E[\overline{\nu}_{X_t}(y+l_t)] - P(\{M_{t-} > y\} \cup \{S_{t-} \leq -l_t\})\overline{\nu}(y+l_t). \end{aligned}$$

From this, (78) and (77), we have that, as $y \rightarrow \infty$,

$$\begin{aligned} P(M > y) &\geq \int_0^\infty E[\overline{\nu}_{X_t}(y+l_t)] dt - [P(M > y) + \varepsilon] \int_0^\infty \overline{\nu}(y+l_t) dt \\ &= (1 + o(1)) \frac{C}{a + \varepsilon} \overline{\nu}^1(y) - (1 + o(1)) [P(M > y) + \varepsilon] \frac{\overline{\nu}^1(y)}{a + \varepsilon} \end{aligned} \quad (79)$$

$$= (1 + o(1)) \frac{C - \varepsilon}{a + \varepsilon} \overline{\nu}^1(y), \quad (80)$$

where (79) follows from Lemma 3.1, and (80) follows since $P(M > y) \rightarrow 0$ as $y \rightarrow \infty$. The required result now follows by letting ε tend to zero. \square

We now derive an asymptotic upper bound for $P(M > y)$ in the case $\kappa < 0$. The proof is similar to that of Lemma 2.3.

Lemma 3.3. *Suppose that (C1)–(C5) hold, that $\nu^1 \in \mathcal{S}$, and that $\kappa < 0$. Then*

$$\overline{\lim}_{y \rightarrow \infty} \frac{P(M > y)}{\overline{\nu}^1(y)} \leq \frac{C}{a}.$$

Proof. For given (small) $\varepsilon > 0$, and (large) $u_0 > 0$, define the linear function

$$u_t := u_0 - (a - \varepsilon)t, \quad t \geq 0. \quad (81)$$

Define the stopping time

$$\sigma := \inf\{t \geq 0 : S_t > u_t\}. \quad (82)$$

Since $S_t/t \rightarrow -a$ a.s., it follows that (for fixed ε),

$$P(\sigma < \infty) \rightarrow 0 \quad \text{as } u_0 \rightarrow \infty. \quad (83)$$

With regard to random variables such as S_σ and $M_\sigma := \max_{0 \leq t \leq \sigma} S_t$ which are only defined on $\{\sigma < \infty\}$, we again make the convention that, for example, $P(M_\sigma > y) := P(M_\sigma > y, \sigma < \infty)$.

We first derive an upper bound for the tail of M_σ . It follows from (81) and (82) that $M_{\sigma-} \leq u_0$ a.s. on $\{\sigma < \infty\}$, and further that, for $y \geq u_0$,

$$P(M_\sigma > y) = P(S_\sigma > y) \leq P(\Delta S_\sigma > y - u_\sigma).$$

Let Φ be the point process whose conditional intensity measure is given by (55). Define also

$$W = \{(t, z) \in \mathbb{R}_+ \times \mathbb{R} : z > y - u_t\} \quad (84)$$

Note that if $\sigma < \infty$ and $\Delta S_\sigma > y - u_\sigma$, then $(\sigma, \Delta S_\sigma) \in W$ and hence the point process Φ has at least one point in the region W . Combining this last observation with the estimate (84), we obtain

$$\begin{aligned} P(M_\sigma > y) &\leq P(\Phi(W) > 0) \\ &= E[P(\Phi(W) > 0 \mid X)] \\ &\leq E[E(\Phi(W) \mid X)] \end{aligned} \tag{85}$$

$$\begin{aligned} &= E \int_0^\infty \overline{\nu}_{X_t}(y - u_t) dt \\ &= \int_0^\infty dt \int_{\mathcal{X}} \pi_t(dx) \overline{\nu}_x(y - u_t) \end{aligned} \tag{86}$$

$$\sim \frac{C}{a - \varepsilon} \overline{\nu}^1(y), \tag{87}$$

as $y \rightarrow \infty$, where (85) follows since $\Phi(W)$ is nonnegative integer-valued, and (87) follows from Lemma 3.1.

Since $\kappa < 0$ it follows from (59) that we can choose any $\alpha \in (0, -\kappa)$ and then $\beta > 0$ sufficiently large that

$$a_x^\beta \leq -\alpha \quad \text{for all } x \in \mathcal{X}. \tag{88}$$

On the set $\{\sigma < \infty\}$ define the process $(S_t^\sigma, t \geq 0)$ by $S_t^\sigma := S_{\sigma+t} - S_\sigma$; let $M^\sigma := \sup_{t \geq 0} S_t^\sigma$. Conditional on the background process X and any finite value of σ , the process $(S_t^\sigma, t \geq 0)$ has independent increments and is generated by the triple $(\nu_{X_{\sigma+t}}, v_{X_{\sigma+t}}^2, a_{X_{\sigma+t}}, t \geq 0)$. Further, it follows from the conditions (C1), (C3), (C4), and (88), that, again conditional on X and σ , this triple satisfies the conditions (62)–(65) of Proposition 3.1 (with ν, α, β as defined here and γ, σ^2 as defined by (C3) and (C4)). It therefore follows from Proposition 3.1 that there exists a constant r , depending on $\nu, \alpha, \beta, \gamma$ and v^2 only, such that, for all x , all finite t and all $y \geq 0$,

$$P(M^\sigma > y \mid X = x, \sigma = t) \leq r \overline{\nu}^1(y); \tag{89}$$

further, conditional on $X = x$ and $\sigma = t$, the random variables M_σ and M^σ are independent. For $y > u_0$, we have $M = S_\sigma + M^\sigma = M_\sigma + M^\sigma$. We now argue exactly as in the proof of Lemma 2.3, starting from the introduction of the random variable \widetilde{M} and with F replaced by ν throughout, to obtain the required result. \square

Proof of Theorem 3.1. This is now immediate from Lemmas 3.2 and 3.3. \square

Proof of Theorem 3.2. This is very similar to, but slightly simpler than, the proof of Theorem 2.2. Let nonnegative $b > \kappa$ be such that the condition (C6) holds. Choose

$$\delta \in (0, \min(a, b - \kappa)) \tag{90}$$

and choose $\varepsilon \in (0, a - \delta)$. Note that, from the condition (C6),

$$\int_{\mathcal{X}} (a_x + \delta) \pi(dx) = -a + \delta < -\varepsilon.$$

It now follows from the definition (59) of κ , and since $b > \kappa$, that we may choose $\beta > 0$ sufficiently large that

$$\int_{\mathcal{X}} (a_x^\beta + \delta) \pi(dx) < -\varepsilon, \tag{91}$$

$$a_x^\beta + \delta \leq b \quad \text{for all } x \in \mathcal{X}. \tag{92}$$

Hence (as for example in the proof of Theorem 2.2) we may define a measurable function b_x on \mathcal{X} such that,

$$a_x^\beta + \delta \leq b_x \leq b, \quad x \in \mathcal{X}, \quad (93)$$

$$\int_{\mathcal{X}} b_x \pi(dx) = -\varepsilon. \quad (94)$$

Define now the processes $(S_t^b, t \geq 0)$ and $(\hat{S}_t, t \geq 0)$ by, for each t ,

$$S_t^b = \int_0^t b_{X_t}, \quad \hat{S}_t = S_t - S_t^b. \quad (95)$$

Note that, conditional on the background process X , the process $(\hat{S}_t, t \geq 0)$ has independent increments and a distribution which is given by the triple $(\nu_{X_t}, v_{X_t}^2, \hat{a}_{X_t}, t \geq 0)$, where, for each x , we have $\hat{a}_x = a_x - b_x$. It follows from (94) that the process $(\hat{S}_t, t \geq 0)$ satisfies the conditions (C1)–(C5) with a is replaced by $a - \varepsilon$. Further, from the definitions (58), (59) and the first inequality in (93), the constant κ associated this process is replaced by some $\hat{\kappa}$ satisfying $\hat{\kappa} \leq -\delta$. Since also $\nu^1 \in \mathcal{S}$, it follows from Lemma 3.3 that the supremum \hat{M} of the process $(\hat{S}_t, t \geq 0)$ satisfies

$$\overline{\lim}_{y \rightarrow \infty} \frac{P(\hat{M} > y)}{\overline{\nu^1}(y)} \leq \frac{C}{a - \varepsilon}. \quad (96)$$

It also follows from the conditions (C1)–(C5) and the first inequality in (93) that the family $(\nu_{X_t}, v_{X_t}^2, \hat{a}_{X_t}, t \geq 0)$ satisfies the conditions (62)–(65) of Proposition 3.1 with α replaced by δ . Hence there exists a constant r such that, for all $x \in \mathcal{X}$, and for all y ,

$$P(\hat{M} > y \mid X = x) \leq \min(1, r\overline{\nu^1}(y)). \quad (97)$$

Now consider the process $(S_t^b, t \geq 0)$. Recall that the condition (C6) corresponds to the discrete-time condition (D4) with F replaced by ν . Recall also that $(T_k, k \geq 0)$ is the sequence of regeneration epochs of the modulating process X . By considering the discrete-time process $(S_{T_k}^b, k \geq 0)$, it follows *exactly* as in the proof of Theorem 2.2 that, under the condition (D7), the supremum M^b of the process $(S_t^b, t \geq 0)$ satisfies

$$\lim_{y \rightarrow \infty} \frac{P(M^b > y)}{\overline{\nu^1}(y)} = 0. \quad (98)$$

Finally, since $M \leq \hat{M} + M^b$, and since \hat{M} and M^b are conditionally independent given X , it follows from (96), (97), (98) and Lemma A.2 that

$$\overline{\lim}_{y \rightarrow \infty} \frac{P(M > y)}{\overline{\nu^1}(y)} \leq \overline{\lim}_{y \rightarrow \infty} \frac{P(\hat{M} + M^b > y)}{\overline{\nu^1}(y)} \leq \frac{C}{a - \varepsilon}. \quad (99)$$

By letting $\varepsilon \rightarrow 0$ in (99) and combining this result with the lower bound given by Lemma 2.2, we now obtain the required result. \square

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A Appendix

In this appendix we give various general results concerning the addition of subexponential random variables. We also justify the generalisations of the Strong Law of Large Numbers given by (4) and (57).

Lemma A.1 below encapsulates the principle of one big jump for subexponential random variables. The result (101) is standard—see, e.g., Baccelli, Schlegel and Schmidt (1999), while the immediately following result follows by standard coupling arguments.

Lemma A.1. *Suppose that $F \in \mathcal{S}$. Let Y_1, \dots, Y_n be independent random variables such that, for each $i = 1, \dots, n$, there exists a constant $c_i > 0$ with*

$$P(Y_i > y) \sim c_i \bar{F}(y) \quad \text{as } y \rightarrow \infty \quad (100)$$

(where in the case $c_i = 0$ this is taken to mean $P(Y_i > y) = o(\bar{F}(y))$ as $y \rightarrow \infty$). Then

$$P(Y_1 + \dots + Y_n > y) \sim (c_1 + \dots + c_n) \bar{F}(y) \quad \text{as } y \rightarrow \infty. \quad (101)$$

Further, if, in (100), “ \sim ” is replaced by “ \lesssim ” for each i , then (101) continues to hold with “ \sim ” similarly replaced by “ \lesssim ”.

The following lemma gives a version of Lemma A.1 (for the case $n = 2$ and with “ \lesssim ”) where the random variables Y_1 and Y_2 are conditionally independent. It requires an extra, asymmetric, condition (which is automatically satisfied in the case of unconditional independence).

Lemma A.2. *Suppose that $F \in \mathcal{S}$. Let Y_1 and Y_2 be random variables which are conditionally independent with respect to some σ -algebra \mathcal{F} and are such that, for some constants $c_1 \geq 0$, $c_2 \geq 0$, and some $r > 0$,*

$$P(Y_i > y) \lesssim c_i \bar{F}(y) \quad \text{as } y \rightarrow \infty, \quad i = 1, 2, \quad (102)$$

$$P(Y_1 > y \mid \mathcal{F}) \leq r \bar{F}(y) \quad \text{for all } y \quad \text{a.s.} \quad (103)$$

(with the case $c_i = 0$ interpreted as in Lemma A.1). Then

$$P(Y_1 + Y_2 > y) \lesssim (c_1 + c_2) \bar{F}(y) \quad \text{as } y \rightarrow \infty. \quad (104)$$

Proof. Since $F \in \mathcal{S}$ implies $F \in \mathcal{L}$, we can choose a positive increasing function h_y of y such that $h_y \rightarrow \infty$ as $y \rightarrow \infty$, but the convergence is sufficiently slow that $\bar{F}(y - h_y) \sim \bar{F}(y)$ as $y \rightarrow \infty$ (see, for example, Foss and Zachary (2003)). The condition $F \in \mathcal{S}$ then implies that

$$\int_{h_y}^{y-h_y} dF(z) \bar{F}(y - z) = o(\bar{F}(y)) \quad \text{as } y \rightarrow \infty. \quad (105)$$

Let Y' be a random variable which is independent of Y_2 and such that

$$P(Y' > y) = 1 \wedge r \bar{F}(y) \quad \text{for all } y. \quad (106)$$

It follows by conditioning on \mathcal{F} , and using the conditional independence of Y_1 and Y_2 with respect to \mathcal{F} , together with (103) and (106), that

$$\begin{aligned} P(h_y < Y_2 < y - h_y, Y_1 + Y_2 > y) &\leq P(h_y < Y_2 < y - h_y, Y' + Y_2 > y) \\ &= o(\bar{F}(y)) \quad \text{as } y \rightarrow \infty, \end{aligned} \quad (107)$$

where (107) follows from (102), (105) and (106). Hence

$$\begin{aligned}
P(Y_1 + Y_2 > y) &= P(Y_2 \leq h_y, Y_1 + Y_2 > y) + P(h_y < Y_2 \leq y - h_y, Y_1 + Y_2 > y) \\
&\quad + P(Y_2 > y - h_y, Y_1 + Y_2 > y) \\
&\lesssim P(Y_1 > y - h_y) + o(\bar{F}(y)) + P(Y_2 > y - h_y) \\
&\lesssim (c_1 + c_2)\bar{F}(y) \quad \text{as } y \rightarrow \infty,
\end{aligned} \tag{108}$$

as required, where (108) follows from (102) and the definition of h_y . \square

Remark A.1. In the case where, in (102), “ \lesssim ” is replaced by “ \sim ”, and where additionally the condition (103) also holds with Y_1 replaced by Y_2 , the conclusion (104) may be strengthened to similarly replace “ \lesssim ” by “ \sim ”. This follows by similar use of the function h_y .

Lemma A.3 below is a variant of a well-known result.

Lemma A.3. *Let Y_1 and Y_2 be independent random variables with distribution functions F_1 and F_2 respectively. Suppose that $F_1^1 \in \mathcal{L}$ and that $Y_2 \geq 0$ a.s. ($F_2(y) = 0$ for $y < 0$). Then the distribution function F of $Y = Y_1 - Y_2$ satisfies*

$$\bar{F}^1(y) \sim \bar{F}_1^1(y) \quad \text{as } y \rightarrow \infty. \tag{109}$$

In particular, $F^1 \in \mathcal{L}$.

Proof. The result is well-known when F_1^1 and F^1 in the statement of the lemma are replaced by F_1 and F respectively—see, e.g., Baccelli, Schlegel, and Schmidt (1999), and the modifications required for the present variation are trivially checked. \square

We now prove the generalisations of the Strong Law of Large Numbers given by (4) and (57).

Consider first the discrete-time case of Section 2. In the case where the modulating process X is stationary (and, by definition, regenerative) then $(\xi_n, n \geq 0)$ is a stationary regenerative sequence and (4) follows from Birkhoff’s theorem (since the invariant σ -algebra is here trivial). In the general case, one can always define a coupling of the sequence $(\xi_n, n \geq 0)$ and of a stationary regenerative sequence $(\xi'_n, n \geq 0)$, such that

$$\xi_{T_1+m} = \xi'_{T'+m} \quad \text{a.s.} \quad \text{for all } m = 1, 2, \dots$$

for some non-negative and a.s. finite integer-valued random variable T' —see, for example, Thorisson (2000, Chapter 10, Section 3.) Therefore, on the event $\{T_1 < n\}$,

$$S_n = S'_{T'-T_1+n} - S'_{T'} + S_{T_1}$$

and, as $n \rightarrow \infty$,

$$\frac{S_n}{n} = \frac{S'_{T'-T_1+n}}{T' - T_1 + n} \frac{T' - T_1 + n}{n} + \frac{S_{T_1} - S'_{T'}}{n} \rightarrow -a \quad \text{a.s.}$$

since the events $\{T_1 < n\}$ increase in n to an event of probability 1.

The continuous-time result (57) follows entirely similarly.

Finally, and for completeness, we give a shortened and simplified proof of Lemma 2.2, which is taken from Foss and Zachary (2002)

Proof of Lemma 2.2. Given $\varepsilon > 0$, by the weak law of large numbers we may choose a constant l_0 sufficiently large that if, for each n , we define $l_n = l_0 + (a + \varepsilon)n$, then

$$P(S_n > -l_n) > 1 - \varepsilon. \quad (110)$$

For any fixed $y \geq 0$ and each $n \geq 1$, define $A_n := \{M_{n-1} \leq y, S_{n-1} > -l_{n-1}, \xi_n > y + l_{n-1}\}$. Since, conditional on the background process X , the random variables ξ_n are independent, it follows that

$$\begin{aligned} P(A_n) &= E[\mathbf{1}(\{M_{n-1} \leq y, S_{n-1} > -l_{n-1}\})\overline{F_{X_n}}(y + l_{n-1})] \\ &\geq E[\overline{F_{X_n}}(y + l_{n-1})] - P(\{M_{n-1} > y\} \cup \{S_{n-1} \leq -l_{n-1}\})\overline{F}(y + l_{n-1}) \end{aligned} \quad (111)$$

$$\geq E[\overline{F_{X_n}}(y + l_{n-1})] - [P(M > y) + \varepsilon]\overline{F}(y + l_{n-1}), \quad (112)$$

where (111) follows from the condition (D1) and (112) follows from (110). Since also the events A_n , $n \geq 1$, are disjoint and each is contained in the event $\{M > y\}$, it follows that

$$\begin{aligned} P(M > y) &\geq \sum_{n \geq 1} E[\overline{F_{X_n}}(y + l_{n-1})] - [P(M > y) + \varepsilon] \sum_{n \geq 1} \overline{F}(y + l_{n-1}) \\ &= (1 + o(1))\frac{C}{a + \varepsilon}\overline{F^{\Gamma}}(y) - (1 + o(1))[P(M > y) + \varepsilon]\frac{\overline{F^{\Gamma}}(y)}{a + \varepsilon} \end{aligned} \quad (113)$$

$$= (1 + o(1))\frac{C - \varepsilon}{a + \varepsilon}\overline{F^{\Gamma}}(y), \quad (114)$$

as $y \rightarrow \infty$, where (113) follows from Lemma 2.1, and (114) follows since $P(M > y) \rightarrow 0$ as $y \rightarrow \infty$. The required result now follows by letting ε tend to zero. \square

References

- [1] ALSMEYER G. AND SBIGNEV, M. (1999). On the tail behaviour of the supremum of a random walk defined on a Markov chain. *Yokohama Math. J.* **46**, 139–159.
- [2] ARNDT, K. (1980). Asymptotic properties of the distribution of the supremum of a random walk on a Markov chain. *Th. Prob. Appl.* **25**, 309–324.
- [3] ASMUSSEN, S. (1999). Semi-Markov queues with heavy tails. In: *Semi-Markov Models and Applications*, J. Janssen and N, Limnios (eds.). Kluwer.
- [4] ASMUSSEN, S. (2000). *Ruin Probabilities*. World Scientific.
- [5] ASMUSSEN, S. AND MØLLER, J.R. (1999). Tail asymptotics for $M/G/1$ type queueing processes with subexponential increments. *Queueing Systems* **33**, 153–176.
- [6] ASMUSSEN, S., SCHMIDLI, H. AND SCHMIDT, V. (1999). Tail probabilities for non-standard risk and queueing processes with subexponential jumps. *Adv. Appl. Probab.*, **31**, 422–447.
- [7] BACCELLI, F. AND FOSS, S. (2004). Moments and tails in monotone-separable stochastic networks. *Ann. Appl. Probab.*, **14**, 612–650.
- [8] BACCELLI, F., FOSS, S. AND LELARGE, M. (2004). Asymptotics of subexponential max-plus networks: the stochastic event graph case. *Queueing Systems*, **46**, 75–96.
- [9] BACCELLI, F., FOSS, S. AND LELARGE, M. (2005). Asymptotics of a maximal dater in generalized Jackson networks. *J. Appl. Probab.*, **42**, 513–530.
- [10] BACCELLI, F., SCHLEGEL, S. AND SCHMIDT, V. (1999). Asymptotics of stochastic networks with subexponential service times. *Queueing Systems* **33**, 205–232.
- [11] BERTOIN, J. (1998). *Lévy Processes*. Cambridge Univ. Press.
- [12] CHISTYAKOV, V.P. (1964). A theorem on sums of independent positive random variables and its applications to branching random processes. *Theory Probab. Appl.*, **9**, 640–648.
- [13] EMBRECHTS, P., KLÜPPELBERG, C. AND MIKOSCH, T. (1997). *Modelling Extremal Events*. Springer-Verlag.
- [14] DENISOV, D., FOSS, S. AND KORSHUNOV, D. (2004). Tail asymptotics for the supremum of a random walk when the mean is not finite. *Queueing Systems*, **46**, 15–33.
- [15] EMBRECHTS, P. AND OMEY, E. (1982). A property of long-tailed distributions. *J. Appl. Probab.*, **21**, 80–87.
- [16] EMBRECHTS, P. AND VERAVERBEKE, N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance Math. Econom.* **1**, 55–72.
- [17] FOSS, S. AND ZACHARY, S. (2002). Asymptotics for the maximum of a modulated random walk with heavy-tailed increments. In: Analytic Methods in Applied Probability (in memory of Fridrih Karpelevich), *American Mathematical Society Translations, Series 2*, **207**, 37–52.

- [18] FOSS, S. AND ZACHARY S. (2003) The maximum on a random time interval of a random walk with long-tailed increments and negative drift. *Ann. Appl. Probab.*, **13**, 37–53.
- [19] HANSEN, N.R. AND JENSEN, A.T. (2005) The extremal behaviour over regenerative cycles for Markov additive processes with heavy tails. *Stoch. Proc. Appl.*, **115** (4), 579–591.
- [20] HUANG, T. AND SIGMAN, K. (1999). Steady-state asymptotics for tandem, split-match and other feedforward queues with heavy-tailed service. *Queueing Systems* **33**, 233–259.
- [21] JELENKOVIĆ, P. AND LAZAR, A. (1998). Subexponential asymptotics of a Markov-modulated random walk with queueing applications. *J. Appl. Prob.* **25**, 132–141.
- [22] KALLENBERG, O. (2002). *Foundations of Modern Probability*. Springer, New York.
- [23] KLÜPPELBERG, C. (1988). Subexponential distributions and integrated tails. *J. Appl. Prob.* **35**, 325–347.
- [24] KLÜPPELBERG, C., KYPRIANOU, A.E. AND MALLER, R.A. (2004). Ruin probabilities and overshoots for general Lévy insurance risk processes. *Ann. Appl. Prob.* **14** (4), 1766–1801.
- [25] KONSTANTOPOULOS, T., LAST, G., AND LIN, S.-J. (2004). Non-product form and tail asymptotics for a class of Lévy stochastic networks. *Queueing Systems* **46**, no. 3-4, 409–437.
- [26] KONSTANTOPOULOS, T. AND RICHARDSON, G. (2002). Conditional limit theorems for spectrally positive Lévy processes. *Adv. Appl. Prob.* **34**, 158–178.
- [27] MAULIK, K. AND ZWART, B. (2005). Tail asymptotics for exponential functionals of Lévy processes. To appear in *Stoch. Proc. Appl.*.
- [28] PAKES, A. (1975). On the tails of waiting time distributions. *J. Appl. Prob.* **7**, 745–789.
- [29] SATO, K.-I. (2000). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Univ. Press.
- [30] SHNEER, V. (2004). Estimates for distributions of sums of random variables with subexponential distributions. *Sibirsk. Mat. Zh.*, **45**, no. 6, 1401–1420.
- [31] THORISSON, H. (2000). *Coupling, stationarity, and regeneration*. Springer-Verlag, New York.
- [32] VERAVERBEKE, N. (1977). Asymptotic behavior of Wiener-Hopf factors of a random walk. *Stoch. Proc. Appl.* **5**, 27–37.
- [33] ZACHARY, S. (2004). A note on Veraverbeke’s theorem. *Queueing Systems*, **46**, 9–14.

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