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ESTIMATES FOR OVERSHOOTING AN ARBITRARY BOUNDARY BY A RANDOM WALK AND THEIR APPLICATIONS*

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Abstract. Estimates are found for the magnitude of overshoot, by a sequence of random variables, over an arbitrary boundary. If the sequence increments satisfy a so-called condition of asymptotic homogeneity and the boundary is asymptotically "smooth," then the occurrence of the weak convergence to a limit shape (as the boundary is sent away) is established for the distribution of the overshoot value. As an application, we obtain a uniform (over the class of distributions) basic renewal theorem and determine the asymptotics of the average time of crossing a curvilinear border by the trajectories of asymptotically homogeneous Markov chains.

Key words. sequence of random variables, Markov chain, random walk, time and value of the first overshoot, uniform integrability, nonlinear boundary, asymptotic homogeneity

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1. Introduction. Consider a sequence $X \equiv \{X(n)\}_{n\geqq 0}$, $X(0)=0$, of random variables with finite expectations. Let $\mathcal{F}_n$ denote the sigma-algebra generated by $X(0), X(1), \dots, X(n)$. For $x \geqq 0$ introduce the time of first crossing the level x

$$\eta(x) = \min\bigl\{k \geqq 1\colon X(k) > x\bigr\} \tag{1.1}$$

and set $\eta(x)=\infty$, whenever $\sup_k X(k) \leqq x$. The overshoot value is defined on the event $\{\eta(x)<\infty\}$ by

$$\chi(x) = X\bigl(\eta(x)\bigr) - x. \tag{1.2}$$

A number of papers (see, e.g., [1], [2], [3], [4], [5], [6], and the references therein) considered a sequence $\{X(n)\}$ of the form

$$X(n) = S(n) + \theta(n), \quad S(n) = \xi_1 + \cdots + \xi_n, \tag{1.3}$$

where $\{\xi_n\}$ are independent identically distributed (i.i.d.) random variables (r.v.'s) with positive expectation and the r.v. $\theta(n)$ meets a number of conditions, in particular, $\theta(n)=o(n)$ (in a sense), and also the sequences $\{(\xi_i,\theta(i))\}_{i\leqq n}$ and $\{\xi_i\}_{i>n}$ are mutually independent for any n. In that case, $X(n)$ is often called a "perturbed" or "nonlinear" renewal process (see, e.g., [4]). Models of this kind arise naturally in problems of sequential analysis and queuing theory, the main objects of investigation being the stopping times $\eta(x)$ and overshoot values $\chi(x)$.

One usually assumes that r.v.'s ξ_n have a finite second moment and r.v.'s $\theta(n)$ admit a representation $\theta(n) = -\gamma(n) - h(n)$, where $\gamma(n)$ form a stochastically bounded sequence and $h(\cdot)$ is a deterministic function, $h(n)=o(n)$. The most general results

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in this setting were obtained in [2] under the hypothesis that h is twice differentiable and, moreover, $h''(x) = O(x^{\alpha-2})$, $h'(x) = O(x^{\alpha-1})$ for some $\alpha \in [0,1)$. Under several additional technical requirements it was shown that

(a) a local renewal theorem is valid for $\sum_{n=1}^{\infty} \mathbf{P}\{X(n) \in (u, u+v]\}$ as $u \to \infty$;

(b) as the boundary is sent away, the distribution of the value of overshoot over it tends to a limiting one;

(c) for the average crossing time the asymptotic distribution (of accuracy $o(1)$) is established.

Set $x + h(n) = g(x,n)$. Then, under the mentioned conditions on $\theta(n)$, one has

$$\eta(x) = \min\bigl\{k \geqq 1\colon S(k) > g(x,k) + \gamma(k)\bigr\},$$
$$\chi(x) = S\bigl(\eta(x)\bigr) - g\bigl(x, \eta(x)\bigr) - \gamma\bigl(\eta(x)\bigr).$$

In this situation the distributions of time and value of overshooting a "constant" boundary x by a sequence $X(n)$ coincide with those for overshooting a nonlinear boundary $g(x,\cdot) + \gamma(\cdot)$ by the sequence of partial sums $S(n)$.

In this paper, the main attention is devoted to studying the distribution of the value of overshoot over a nonlinear boundary $g(x,\cdot) + \gamma(\cdot)$ by a sequence $\{X(n)\}$ from a more general class, whose increments $X(n+1) - X(n)$ need not be equidistributed, need not have finite second moment, and can be dependent. In addition, the representation $x + h(n)$ for the function $g(x,n)$ in general might fail. It turns out that in broad assumptions very precise estimates hold for $\chi(x)$ and, moreover, the statements on weak convergence of the overshoot value are valid.

At first we consider the case of "constant" boundary x (i.e., $g(x,n) \equiv x$, $\gamma(n) \equiv 0$ with x increasing to infinity) and demonstrate (see Theorem 2.1) that, under certain conditions (see (G1)–(G3)), the average time of the first crossing cannot grow faster than a linear in x function. In this case the distributions of the overshoot values admit nonimprovable estimates. Next we turn to investigating the case of a Markov chain $\{X(n)\}$ (in general, nonhomogeneous) having asymptotically homogeneous jumps (with the growth of spatial and temporal variables). Assuming that the distribution of jump "at infinity" is nonlattice, we prove the weak convergence (as $x \to \infty$) of the distributions of overshoot values to the limit (see Theorem 2.2), which is the distribution of overshooting an infinitely distant level by a homogeneous random walk. Then the last statement is carried over to the case of nonlinear deterministic (Theorem 2.3) and random (Theorem 2.4) boundaries. Further, it is shown that in general the limit distribution of overshoot value can have a different form (Theorem 2.5). Along with Markov chains, one may consider in Theorems 2.2–2.5 sequences $X(n)$ of a more general type.

We obtain as corollaries uniform renewal theorems in the "homogeneous" (Theorems 2.6 and 2.7) and Markovian (Theorem 2.8) cases. After that we give the corollaries for the first crossing time (Theorem 2.9). They are employed in studying two Markov chains of special form which play important roles in applications.

All the proofs of our theorems are given in section 3.

2. Main results and their corollaries.

2.1. The statements on the value of overshoot. Let $X \equiv \{X(n)\}_{n \geqq 0}$, $X(0) = 0$, be an arbitrary sequence of r.v.'s and $\mathcal{F}_n = \sigma(X(0), X(1), \ldots, X(n))$. For any integer $l \geqq 0$, set

$$X^{(l)}(n) = X(l+n) - X(l).$$

We will need the following conditions:

(G1) *There exist a constant $A > 0$ and an integer $m \geqq 1$ such that, for all integers $n \geqq 0$,*

$$\mathbf{E}(X^{(mn)}(m) \mid \mathcal{F}_{mn}) \geqq A \quad a.s. \tag{2.1}$$

Condition (G1) says that the sequence $X^{(mn)} - An$, for $n = 0, 1, \dots,$ forms a submartingale.

(G2) *There exists a distribution F on $[0, \infty)$ such that, for any n and x,*

$$\mathbf{P}\bigl\{X(n+1) - X(n) \geqq x \mid \mathcal{F}_n\bigr\} \leqq F([x, \infty)) \equiv \overline{F}(x) \quad a.s. \tag{2.2}$$

A random variable ζ with distribution F has the finite expectation

$$b = \mathbf{E}\,\zeta \equiv \int_0^\infty \overline{F}(x)\,dx. \tag{2.3}$$

(G3) *If $m > 1$, assume additionally the existence of a constant $V \geqq 0$ such that, for all $k = 0, 1, \dots$ and $j = 1, 2, \dots, m-1$, the following inequality holds a.s.:*

$$\mathbf{E}\bigl(X^{(km+j)}(m-j) \mid \mathcal{F}_{km+j}\bigr) \geqq -V. \tag{2.4}$$

Note that if the increments $\{X(n+1) - X(n);\, n = 0, 1, \dots\}$ form a sequence of independent random variables, then (G3) follows from conditions (G1) and (G2) when $V = mb - A$. Indeed, $\mathbf{E}X^{(km+j)}(m-j) = \mathbf{E}X^{(km)}(m) - \mathbf{E}X^{(km)}(j) \geqq A - jb \geqq A - mb$.

The following statement is true for the variables $\eta(x)$ and $\chi(x)$ defined by (1.1) and (1.2).

THEOREM 2.1. *Given validity of conditions* (G1)–(G3),

(a) *there exists a constant K such that, for any $x \geqq 0$,*

$$\mathbf{E}\,\eta(x) \leqq K(1 + x); \tag{2.5}$$

(b) *there exist constants c, c_1 such that, for any x, $t \geqq 0$, one has*

$$\mathbf{P}\bigl\{\chi(x) > t\bigr\} \leqq c\left(\overline{F}(t) + \int_t^{t+x} \overline{F}(u)\,du\right), \tag{2.6}$$

$$\mathbf{P}\bigl\{\chi(x) > t\bigr\} \leqq c_1 \mathbf{E}\bigl\{\zeta \mathbf{I}(\zeta > t)\bigr\}. \tag{2.7}$$

Remark 2.1. As will be seen from the proof, the constants K, c, and c_1 depend only on A, m, V, and the function $b(t) \equiv \mathbf{E}\{\zeta \mathbf{I}(\zeta > t)\}$. In particular, if $m = 1$, one may set, for any $\varepsilon \in (0, 1)$,

$$c = K = \frac{b^*(\varepsilon A)}{A(1 - \varepsilon)}, \tag{2.8}$$

$$c_1 = \frac{K}{\beta} \max\left(1,\ \beta + \frac{4N}{A}\right), \tag{2.9}$$

where $b^*(\varepsilon) = \min\{t\colon b(t) \leqq \varepsilon\}$, $\beta = \frac{1}{2}\min\{N, 1, A/4\}$, and $N = b^*(A/2)$.

All of the further results in this section can be proved under the conditions (G1)–(G3). However, to simplify the proofs, we set $m = 1$ in condition (G1) and assume in the sequel that the sequence X forms a Markov chain (need not be homogeneous): for $n \geqq 0$

$$X(n+1) = X(n) + \xi\bigl(n+1,\ X(n)\bigr),$$

where the families of r.v.'s $\{\xi(1,x)\}, \{\xi(2,x)\}, \dots$ are mutually independent and, for each n, the r.v. $\xi(n,x) \equiv \xi(n,x,\omega)$ is a measurable function of (x,ω).

Under these assumptions one can rewrite conditions (G1), (G2) as follows:

(MC1) *There exists a constant* $A > 0$ *such that, for any* n *and* x,

$$\mathbf{E}\left|\xi(n,\,x)\right| < \infty \quad \textit{and} \quad \mathbf{E}\xi(n,\,x) \geqq A;$$

(MC2) *There exists a distribution* F *such that, for any* n, x, *and* t,

$$\mathbf{P}\bigl\{\xi(n,\,x) \geqq t\bigr\} \leqq \overline{F}(t)$$

and the mean $b = b(0)$ *in* (2.3) *is finite.*

Let us introduce the condition of "asymptotic homogeneity (AH)":

(AH) *There exists a nonlattice random variable* ξ *such that*

(1) $\mathbf{E}\,\xi$ *is finite*;

(2) *the distributions of the random variable* $\xi(n,x)$ *are weakly convergent to that of* ξ *as* $n,x \to \infty$.

The last requirement means that $\rho(\xi(n,x),\xi) \to 0$ as $n,x \to \infty$, where ρ is the Lévy metric (F_ξ being the distribution of ξ):

$$\rho(\xi_1,\xi_2) \equiv \rho(F_{\xi_1},F_{\xi_2}) = \inf\bigl\{\varepsilon > 0\colon \forall\, x,\ F_{\xi_i}(x) \leqq F_{\xi_j}(x+\varepsilon)+\varepsilon,\ i,j = 1,\,2\bigr\}.$$

We interpret the nonlatticity of the r.v. ξ as the following property: for any $C > 0$, $\sum_{n=-\infty}^{\infty} \mathbf{P}\{\xi = nC\} < 1$.

Remark 2.2. A number of results below remain valid for the lattice case after a natural modification of the statements.

Remark 2.3. If a family of r.v.'s $\{\xi(n,x)\}$ meets conditions (MC1), (MC2), and (AH), then (MC1) and (MC2) also hold for ξ with the same A and F.

Introduce an auxiliary sequence of i.i.d. r.v.'s $\{\xi_n\}_{n\geqq 1}$, where $\xi_1 \overset{\mathrm{d}}{=} \xi$ ($\overset{\mathrm{d}}{=}$ denotes here the equality of distributions). For the random walk $S(0) = 0$, $S(n) = \xi_1+\dots+\xi_n$, set

$$\eta_{as}(x) = \min\{n \geqq 1\colon S(n) \geqq x\}, \qquad \chi_{as}(x) = S(\eta_{as}(x)) - x.$$

As is known, in the nonlattice case there exists an r.v. $\chi_{as}(\infty)$ (a so-called overshoot over the infinitely distant level) such that $\rho(\chi_{as}(x),\chi_{as}(\infty)) \longrightarrow 0$ as $x \to \infty$. Note that $\chi_{as}(\infty)$ has absolutely continuous distribution. For $x \geqq 0$, let $\rho(x) = \sup_{y \geqq x} \rho(\chi_{as}(y),\chi_{as}(\infty))$, and hence $\rho(x) \to 0$ as $x \to \infty$.

For $y > 0$ set $\rho_{N,y} = \sup_{m\geqq N; x \geqq y} \rho(\xi(m,x),\xi)$ and $N(y) = [y^{2/3}]$ (here $[x]$ stands for the integer part of x). Further, we adopt the notation $z \equiv z(y) = \{\min y, \rho^{-1}_{N(y),y}\}^{1/3}$.

THEOREM 2.2. *Assume that* $\{X(n)\}$ *is a Markov chain satisfying conditions* (MC1), (MC2), *the r.v.* ξ *has a nonlattice distribution*, *and let a number* $y > 0$ *be taken so that* $z \equiv z(y) \geqq 1$. *Then*, *for any* $x \geqq 2y + 2z(y) + 1$, *one has*

$$\rho\bigl(\chi(x),\ \chi_{as}(\infty)\bigr) \leqq \frac{3(K+b)+1}{z(y)} + c_1 b(z) + \rho(z), \tag{2.10}$$

where $\chi(x)$ *is defined in* (1.2), $b(t)$ *in Remark* 2.1, *and* $b = b(0)$. *Consequently, under condition* (AH),

$$\rho\big(\chi(x),\ \chi_{as}(\infty)\big) \longrightarrow 0 \quad \text{as } x \to \infty. \tag{2.11}$$

Inequality (2.10) shows that the values of overshoot $\chi(x)$ converge to $\chi_{as}(\infty)$ uniformly in a certain class of Markov chains. More precisely, the following is valid.

COROLLARY 2.1. *Let* $B(t)$ *and* $R(y, N)$ *be nonincreasing functions such that* $\lim_{t\to\infty} B(t) = \lim_{y,N\to\infty} R(y, N) = 0$ *and let the r.v.* ξ *have a nonlattice distribution with a finite positive mean. Consider the class* $\mathcal{X}$ *of Markov chains* $X \equiv \{X(n)\}$ *which satisfies conditions* (MC1), (MC2), *and* (AH) *with given* A *and* ξ *and assume the validity of inequalities* $b(t) = \mathbf{E}\{\zeta \mathbf{I}(\zeta > t)\} \leqq B(t)$ *and* $\rho_{y,N} \leqq R(y, N)$ *for all* t, y, N (ξ *being defined in* (2.3)).

Then

$$\sup_{X\in\mathcal{X}} \rho(\chi(x),\ \chi_{as}(\infty)) \longrightarrow 0 \quad \text{as } x \to \infty. \tag{2.12}$$

In particular, relation

$$\sup_{l} \sup_{y\geqq x} \rho(\chi^{(l)}(y), \chi_{as}(\infty)) \longrightarrow 0 \quad \text{as } x \to \infty \tag{2.13}$$

holds for any Markov chain $X = \{X(n)\}$ *satisfying* (MC1), (MC2), *and* (AH). *Here* $\chi^{(l)}$ *denotes the value of the first overshoot over the level* x *for the sequence* $\{X^{(l)}(n),\ n = 0, 1, \ldots\}$.

Theorem 2.2 can be extended to the case of nonlinear boundaries as follows.

Consider a family of functions $g(x, \cdot)\colon \mathbf{R}^+ \to \mathbf{R}^+$, $x > 0$, with $g(x, 0) = x$. We will employ the values $g(x, n)$ at integer $n = 1, 2, \ldots$ only. Therefore, without loss of generality it may be assumed that all $g(x, t)$ are continuous in t and $\min(g(x, [t]), g(x, [t] + 1)) \leqq g(x, t) \leqq \max(g(x, [t]), g(x, [t] + 1))$ for any x and t.

Let

$$\eta_g(x) = \min\big\{n \geqq 0\colon X(n) > g(x,\ n)\big\} \tag{2.14}$$

be the time of the first overshoot over the boundary $g(x, \cdot)$ and $\chi_g(x)$ the corresponding overshoot value.

Fix $\Delta > 0$. Let $U \equiv U(x)$ be the minimal solution of the equation $g(x, t) = (b + \Delta)\, t$, where b appears in (2.3). Suppose that the choice of $g(x, \cdot)$ and Δ ensures that $U(x) < \infty$ for all $x > 0$, along with the validity of conditions

$$\liminf_{x\to\infty} \frac{1}{x} \inf_{t\in[0,U(x)]} g(x,\ t) > 0; \tag{g1}$$

(g2) *for any* $T > 1$,

$$\sup_{t\in[U,TU]} \big|g(x,\ t) - g(x,\ t + 1)\big| \longrightarrow 0$$

as $x \to \infty$, *where* A *is the same as in* (G1) *and* (MC1).

Instead of (g2) one might consider a seemingly more general condition

$$\sup_{t\in[U,TU]} \big|g(x,\ t) - g(x,\ t + 1) + C\big| \longrightarrow 0$$

for some $C < A$. However, it reduces to (g2) by means of substituting $X(n)$ by $X(n) - nC$ in the initial problem.

Remark 2.4. If the function g has a representation $g(x,t) = x + h(t)$, then conditions (g1), (g2) are valid if and only if $\lim_{n\to\infty} \sup_{m\geqq n} |h(m+1) - h(m)| = 0$. The examples of functions $h(t)$ meeting these conditions are provided by $\log t$; t^α, $\alpha \in (0,1)$; $t/\log t$. Note that $t/\log t$ does not belong to the class of functions considered in [2].

THEOREM 2.3. *Under assumptions* (g1) *and* (g2) *the validity of conditions* (MC1), (MC2), *and* (AH) *implies*

$$\rho\bigl(\chi_g(x),\ \chi_{as}(\infty)\bigr) \longrightarrow 0 \quad \text{as } x \to \infty. \tag{2.15}$$

COROLLARY 2.2. *If, in addition, the r.v.* ζ *has a finite second moment, the uniform in* x *integrability of* $\chi(x)$ *and* $\chi_g(x)$ *ensues and, consequently,* $\mathbf{E}\chi(x) \to \mathbf{E}\chi_{as}(\infty) < \infty$ *and* $\mathbf{E}\chi_g(x) \to \mathbf{E}\chi_{as}(\infty)$ *as* $x \to \infty$.

Remark 2.5. Having assumed that the strong law of large numbers holds for the Markov chain $X(n)$, i.e., $X(n)/n \to a \equiv \mathbf{E}\xi$ a.s. (which is the case if $\{\xi(n) \equiv \xi(n,x)\}$ form a sequence of i.i.d. r.v.'s with the mean a, or, more generally, if $\xi(n,x)$ converge to ξ fast enough), one establishes that the assertion of Theorem 2.3 remains valid if in (g1), (g2) the function $U \equiv U(x)$ is changed for $(a - \varepsilon_x)x$ and TU, respectively, for $(a + \varepsilon_x)x$, where $\varepsilon_x > 0$ are such that $\mathbf{P}\{|X(n)/n - a| > \varepsilon_x\} \longrightarrow 0$ as $x \to \infty$.

Now note a simple corollary to Theorems 2.1 and 2.3.

COROLLARY 2.3. *Let* φ: $(-\infty,\infty) \longrightarrow (-\infty,\infty)$ *be a strictly monotone function such that*

$$\bigl|\varphi(x) - x\bigr| \longrightarrow 0 \quad \text{as } x \to \infty. \tag{2.16}$$

If the Markov chain $\{\varphi(X(n))\}$ meets conditions (MC1) and (MC2), then (2.5) holds for $X(n)$.

If $\{\varphi(X(n))\}$ meets conditions (MC1), (MC2), and (AH), then (2.11) is valid. If, in addition, assumptions (g1), (g2) are fulfilled, then one has (2.15).

Let $\{\gamma(n)\}_{n\geqq 1}$ be a sequence of r.v.'s such that, for any n, $\gamma(n)$ is measurable with respect to the σ-algebra generated by r.v.'s $X(1),\dots,X(n)$ (or, more generally, $\gamma(1),\dots,\gamma(n)$ and families $\{\xi(n+1,x)\}_{x\in\mathbf{R}}$, $\{\xi(n+2,x)\}_{x\in\mathbf{R}},\dots$ are mutually independent). Set $\eta_{(\gamma)}(x) = \min\{n \geqq 1\colon X(n) > x + \gamma(n)\}$ and $\chi_{(\gamma)}(x) = X(\eta_{(\gamma)}(x)) - x - \gamma(\eta_{(\gamma)}(x))$.

THEOREM 2.4. *Suppose that* $\{\gamma(n)\}$ *converges a.s. to a finite r.v.* γ. *If a Markov chain* $X(n)$ *satisfies conditions* (MC1), (MC2), *and* (AH), *then*

$$\rho\Bigl(\chi_{(\gamma)}(x),\, \chi_{as}(\infty)\Bigr) \longrightarrow 0 \quad \text{as} \quad x \to \infty. \tag{2.17}$$

In particular, if $\widehat{\gamma} \equiv \sup_n |\gamma(n)|$ *is an integrable r.v. and the r.v.* ζ (*appearing in* (G2) *and* (MC2)) *is square integrable, then*

$$\mathbf{E}\chi_{(\gamma)}(x) \longrightarrow \mathbf{E}\chi_{as}(\infty) \quad \text{as } x \to \infty. \tag{2.18}$$

Theorem 2.4 admits a natural extension to cover the case of arbitrary boundaries $g(x,n)$.

Theorem 2.5 given below demonstrates that, in general, the limit distribution in Theorems 2.2–2.4 may be different from the distribution of $\chi_{as}(\infty)$. For the sake of simplicity we formulate and prove this theorem only in the case of a homogeneous random walk $X(n) = S(n) = \xi_1 + \cdots + \xi_n$ and a nonlinear boundary of a special form.

THEOREM 2.5. *Assume that* $\{\xi_n\}$ *is a sequence formed by i.i.d. r.v.'s with the mean* $a = \mathbf{E}\xi_1 > 0$ *and the finite variance* $\sigma^2 = \mathbf{E}\xi_1^2 - a^2 > 0$.

Let $g(x,t)$ *be representable as* $x + \alpha_1 t$ *for* $t < t_0 \equiv t_0(x) = x/(a - \alpha_1)$ *and* $g(x,t) = ax/(a-\alpha_1) + (t-t_0)\alpha_2 t$ *for* $t \geqq t_0$, *with* α_1, $\alpha_2 \in [0,a)$ *being some constants, and, moreover, let r.v.'s* $\xi_1 - \alpha_1$ *and* $\xi_1 - \alpha_2$ *have nonlattice distributions.*

Then

$$\mathbf{P}\{\chi_g(x) > t\} \longrightarrow \frac{1}{2}\left(\mathbf{P}\{\chi_{as}^{(1)} > t\} + \mathbf{P}\{\chi_{as}^{(2)} > t\}\right),$$

where $\chi_{as}^{(i)} \equiv \chi_{as}^{(i)}(\infty)$ *is the value of overshoot over the infinitely distant barrier by the random walk with increments* $\{\xi_n - \alpha_i\}_{n\geqq 1}$, $i = 1, 2$.

It is not difficult to extend the assertion of Theorem 2.5. to the case of Markov chains and functions g from a wider class. This results in the limit distribution which is a mixture of distributions of overshoots over the infinitely distant barrier for homogeneous random walks with "shifted" increments $\{\xi_n - \alpha\}_{n\geqq 1}$, the mixing being due to the standard normal law.

2.2. The uniform convergence theorem for overshoots and the uniform renewal theorem. It is worth noting that, in view of results of the Wald identity kind, there is a close connection between the statements concerning the existence of the limit distribution $\chi(x)$ for the sequence $X(n) = S(n)$ and the renewal theorem. One of the related statements implies the other (see, e.g., [7]). In this paper, we have chosen the statements on overshoot to be primary. Basing it on them, we establish the renewal theorem below. But since we succeeded in giving the required results on the value of overshoot as statements uniform over the class of distributions, the same uniformity occurs in the renewal theorem.

There exists another approach, equally natural, where one takes as primary the estimates in the renewal theorem and employs them to establish statements on the overshoot value. The corresponding results will be published soon.

Let $B(t)$ be an arbitrary nonnegative function, $\lim_{t\to\infty} B(t) = 0$, and $\psi(u)$, $u > 0$, be a strictly positive function. Introduce the class of distributions $\mathfrak{F} \equiv \mathfrak{F}(A, B, \psi)$. We will say that F belongs to the class $\mathfrak{F}$ if the r.v. ξ with distribution F meets the following conditions:

(i) $\mathbf{E}\,\xi \equiv a(F) \geqq A$;

(ii) $\mathbf{E}\,\{\xi \mathbf{I}(\xi > t)\} \leqq B(t)$ *for all* $t \geqq 0$;

(iii) $|\mathbf{E}\exp\{iu\xi\} - 1| \geqq \psi(u)$ *for all* $u > 1$.

Condition (iii) means that any distribution $F \in \mathfrak{F}$ is nonlattice and any weak limit of distribution functions from $\mathfrak{F}$ is nonlattice as well. Denote by $\chi^{(F)}(x)$ the value of overshoot above the level x for the sequence of sums $S^{(F)}(n) = \xi_1^{(F)} + \cdots + \xi_n^{(F)}$ of i.i.d. r.v.'s having the common distribution F. Let $H^{(F)}(u,v) = \sum_n \mathbf{P}\{S^{(F)}(n) \in (u, u+v]\}$ be the corresponding renewal function.

THEOREM 2.6. 1. *The following relation is valid*:

$$\limsup_{x\to\infty}\sup_{F\in\mathfrak{F}} \rho(\chi^{(F)}(x), \chi^{(F)}(\infty)) = 0. \tag{2.19}$$

2. *If condition* (ii) *in the definition of* $\mathcal{F}$ *is changed for the requirement of* "*two-sided*" *uniform integrability*

$$\text{(2.20)} \quad \text{(ii)}' \qquad \sup_{F\in\mathfrak{F}} \mathbf{E}\Big\{|\xi|\mathbf{I}\big(|\xi|>t\big)\Big\} \leqq B(t),$$

then, for any $v>0$,

$$\text{(2.21)} \qquad \limsup_{u\to\infty} \sup_{F\in\mathfrak{F}} \left| H^{(F)}(u,\,v) - \frac{v}{a(F)} \right| = 0.$$

Note that the additional condition (ii)$'$ is necessary for establishing the following result: if $F^{(n)} \in \mathfrak{F}$ weakly converge to F, then $a(F^{(n)}) \to a(F)$ (whenever all the summands ξ_n are nonnegative, conditions (ii) and (ii)$'$ coincide). After this paper was prepared for publication, we learned that statement (2.21) had already been obtained in a recent paper [2]. The method of the proof there, based on the apparatus of characteristic functions, is different from ours.

It is not difficult to formulate a natural analogue of Theorem 2.6 in the lattice case. To be exact, we have the following theorem.

THEOREM 2.7. *Let* $\psi_1(u),\ u\in[0,2\pi]$, *be a continuous function such that* $\psi_1(0)=\psi_1(2\pi)=0$ *and* $\psi_1(u)>0$ *for all* $u\in(0,2\pi)$. *Furthermore, let* $\mathfrak{F}_1\equiv\mathfrak{F}_1(C,B,\psi_1)$ *be the class of distribution functions which satisfy* (i), (ii), *and the following conditions*:

(iv) *Every* $F\in\mathfrak{F}_1$ *is a distribution function of an integer-valued random variable* ξ *with step* 1 (*i.e.*, $2\pi=\min\{u>0\colon \mathbf{E}\exp\{iu\xi\}=1\}$);

(v) *for an r.v.* ξ *with a distribution* $F\in\mathfrak{F}_1$ *and all* $u\in(0,\,2\pi)$

$$\big|\mathbf{E}\exp\{iu\xi\}-1\big| \geqq \psi_1(u).$$

Then, for any $t=0,1,\ldots,$ *one has*

$$\limsup_{x\to\infty} \sup_{F\in\mathfrak{F}_1} \Big| \mathbf{P}\big\{\chi^{(F)}(x)>t\big\} - \mathbf{P}\big\{\chi^{(F)}(\infty)>t\big\} \Big| = 0,$$

where x *takes integer values.*

Having changed (ii) *for condition* (ii)$'$, *we obtain for* $x\in\{1,2,\ldots\}$

$$\limsup_{x\to\infty} \sup_{F\in\mathfrak{F}_1} \left| \sum_n \mathbf{P}\big\{S(n)=x\big\} - \frac{1}{a(F)} \right| = 0.$$

On account of Theorem 2.2 and Remark 2.1 one can prove natural analogues of the uniform theorem on convergence of overshoots and the uniform renewal theorem for asymptotically homogeneous Markov chains.

Let us formulate one of such statements. Given nondecreasing functions $B(t)\to 0$, $t\to\infty$, and $\{R(n,x)\}$, $R(n,x)\to 0$ as $n,x\to\infty$, a Markov chain $X=\{X(n)\}$ with increments $\{\xi(n,x)\}$ is called (B,R)-*regular* when

(a) it satisfies (MC1), (MC2), and also (AH) with a given estimate of convergence rate $\rho(\xi(n,x),\xi)\leqq R(n,x)$;

(b) the distribution of the majorant ζ in (2.3) satisfies condition

$$b(t)=\mathbf{E}\big\{\zeta\mathbf{I}(\zeta>t)\big\}\leqq B(t) \quad \text{for all } t>0;$$

(c) the distribution of r.v. ξ belongs to the class $\mathfrak{F}$.

THEOREM 2.8. *Let* $\mathcal{X}_1$ *be the family of all* (B,R)-*regular Markov chains. Then*

$$\limsup_{x\to\infty} \sup_{X\in\mathcal{X}_1} \rho\big(\chi(x),\ \chi_{as}(\infty)\big)=0.$$

2.3. Corollaries for the mean time of the first crossing and examples. In this subsection we restrict our attention to the functions $g(x,t) = x + h(t)$. First, note the existing close connection between the statements on the limit distribution $\chi_g(x) \equiv \chi_{(h)}(x)$ and the asymptotics of the average time of the first crossing $\eta = \eta_g(x) \equiv \eta_{(h)}(x)$. This is an implication of the identity

$$X(\eta) = x + h(\eta) + \chi_{(h)}(x).$$

Therefore, when convergence in the mean of $\chi_{(h)}(x)$ to χ occurs and the asymptotics of $\mathbf{E}\, h(\eta)$ and $\mathbf{E}\, X(\eta)$, estimated in terms of $\mathbf{E}\,\eta$, is at our disposal, we deduce an equation for $\mathbf{E}\,\eta$.

For example, if h is an upward convex function and $X(n) = S(n)$, the Wald identity yields

$$x + h(\mathbf{E}\,\eta) + \mathbf{E}\,\chi_{(h)}(x) \leqq a\mathbf{E}\,\eta + o(1).$$

Thus, if $u(x)$ is a solution of the equation $x + h(u) + \mathbf{E}\,\chi = au$, then $\mathbf{E}\,\eta_{(h)}(x) \leqq u(x) + o(1)$. It can be shown that in broad assumptions the inverse inequality is also valid. We will not dwell on this point since similar results for smooth h were obtained in [2].

Here we consider in detail only a special case playing an important role in applications. Namely, we assume as before that $g(x,n) = x + h(n)$ and, moreover,

(MC3) $X(n)$ is a Markov chain having representation $X(n) = S(n) + \theta(n)$, where $S(n) = \xi_1 + \cdots + \xi_n$, the sequence $\{\xi_n\}$ consists of i.i.d. r.v.'s with the mean $\mathbf{E}\,\xi_1 = a > 0$ and the finite second moment $\mathbf{E}\,\xi_1^2 < \infty$, and $\theta(n)$ is measurable with respect to the σ-algebra $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$, $\theta(n) \xrightarrow{p} \theta$, $\mathbf{E}\,\theta^2(n) < C < \infty$.

In the sequel we give two examples of Markov chains, often appearing in applications, that satisfy condition (MC3).

We assume the function h to be smooth upward convex and admitting the representation $h(t) = t^{\alpha} l(t)$, $\alpha < 1$, where $l(t)$ is a slowly variable function "twice differentiable at infinity." Namely, we introduce the following condition:

(g3) For all t, v; $t \geqq 0$, $t + v \geqq 0$, $h(t)$ is representable in the form

$$h(t+v) = h(t)\left[1 + \frac{v}{t}\, c(t) + \frac{v^2}{2t^2}\, c(t,v)\right], \tag{2.22}$$

where $c(t), c(t,v)$ are bounded functions and $c(t) \to \alpha < 1$ as $t \to \infty$.

It should be noted that $h(t) = t^{\alpha} l(t)$ meets condition (g3) whenever $l(t)$ is smooth at infinity (say, $l(t) = \log t$). If $\alpha < \frac{1}{2}$, we do not need the last term in decomposition (2.22), and it suffices to have representation of the form $h(t+v) = h(t)[1 + vc(t,v)/t]$.

From now on we denote by $u(x)$ the solution of equation

$$au = x + h(u) + d,$$

where $d = \mathbf{E}\,\chi - \mathbf{E}\,\theta$ and χ has distribution $\chi_{as}(\infty)$. Such a solution always exists and is unique if $au - h(u)$ is increasing in u (the last assumption might be superfluous in view of (g3)).

THEOREM 2.9. *Given the validity of* (MC3), (g3), *one has*

$$\mathbf{E}\,\eta(x) = u(x) + o(1).$$

Consider a pair of examples of recursive sequences $X(n+1) = f(X(n), \xi_{n+1})$, $f(\cdot,\cdot)\colon \mathbf{R}^2 \to \mathbf{R}$, which play an important role in the theory of queuing service systems and in the problems of sequential analysis (see, e.g., [9]). If ξ_n are i.i.d. r.v.'s, $X(n)$ form a Markov chain.

Example 1. Consider a stochastically recursive sequence

$$w_{n+1} = (w_n + \xi_{n+1})^+ \equiv \max(0,\, w_n + \xi_{n+1})$$

and set, for simplicity, the initial value $w_0 = 0$.

Put $\xi(n,x) = (x+\xi_n)^+ - x^+$. Then $w_{n+1} = w_n + \xi(n+1, w_n)$.

Observe that $\xi_n \leqq \xi(n,x) \leqq \xi_n^+$ a.s. and $\xi(n,x) \to \xi_n$ as $x \to \infty$. So conditions (MC1), (MC2), and (AH) are fulfilled with $A = a$, $\zeta = \xi_1^+$, and $\xi = \xi_1$. Hence, the assertions of Theorems 2.2–2.4 are valid for the sequence $\{w_n\}$.

Further, note the representation $w_n = \max(0, \xi_n,\, \xi_n + \xi_{n-1}, \dots, \xi_n + \dots + \xi_1)$. Therefore, the sequence $\theta(n) \equiv w_n - S(n) = \max(0, -S(1), \dots, -S(n))$ converges monotonically to $\theta \equiv \sup(0, -S(1), -S(2), \dots)$.

The additional moment assumption $\mathbf{E}\,\xi_1^2 < \infty$, $\mathbf{E}\,(\xi_1^-)^3 < \infty$, yields the fulfillment of condition (MC3) and, therefore, the assertion of Theorem 2.9.

We mention that, given the existence of $\mathbf{E}\,\theta$, $\mathbf{E}\,\xi_1^2$, the conditions $\mathbf{E}\,\theta^2 < \infty$, $\mathbf{E}\,(\xi_1^-)^3 < \infty$ are, apparently, superfluous.

Example 2. Consider a stochastically recursive sequence (see [9])

$$W_{n+1} = \xi_{n+1} + \log\bigl(1 + \exp(W_n)\bigr).$$

As in the first example, we assume $W_0 = 0$. Note that

$$\max(0,\, W_n) + \xi_{n+1} < W_{n+1} < \max(0,\, W_n) + \xi_{n+1} + 1 \tag{2.23}$$

a.s. for all n.

Let $\varphi(x) = x$ for $x \geqq 0$ and $\varphi(x) = \exp(x) - 1 \geqq x$ for $x < 0$. The function φ is continuous, strictly increasing, and satisfies the conditions of Corollary 2.3. Set $W_n^* = \varphi(W_n)$ and, for $x > -1$,

$$\xi(n,x) = \varphi\Bigl(\xi_n + \log\bigl(1 + \exp\bigl\{\varphi^{-1}(x)\bigr\}\bigr)\Bigr) - x.$$

For $x \leqq -1$, set $\xi(n,x) = \xi_n$. Then $W_{n+1}^* = W_n^* + \xi(n+1, W_n^*)$ a.s. If $x \gg 1$, then $\xi(n,x) = \xi_n + O(\exp\{-x\})$ a.s. on the event $\{\xi_n \geqq -x\}$. Therefore, $\xi(n,x) \to \xi_n$ a.s. as $x \to \infty$. Note that $\xi_n \leqq \xi(n,x) \leqq \xi_n^+ + 2$ a.s. for any x. Hence, for the Markov chain $\{W_n^*\}$, conditions (MC1), (MC2), and (AH) hold with $A = a$, $\zeta = \xi_1^+ + 2$, and $\xi = \xi_1$. So we can use Corollary 2.3 to conclude that the assertions of Theorems 2.2–2.4 hold for the sequence $\{W_n\}$.

We note further the relations

$$\exp(W_n) = \exp\bigl(S(n)\bigr)\Bigl[1 + \exp\bigl(-S(1)\bigr) + \dots + \exp\bigl(-S(n-1)\bigr)\Bigr].$$

Thus, the sequence

$$\theta(n) \equiv W_n - S(n) = \log\left(\sum_{i=0}^{n-1} \exp(-S(i))\right)$$

converges monotonically to $\theta \equiv \log(\sum_0^\infty \exp(-S(i))) < \infty$. It is not difficult to see that if the second moment $\mathbf{E}\,\xi_1^2$ is finite, then also $\mathbf{E}\,\theta^2 < \infty$. Hence, we arrive at the assertion of Theorem 2.9.

3. Proofs.

3.1. Proofs of the statements of subsection 2.1. We shall need several auxiliary propositions. Lemma 3.1 proves that, for any $x > 0$, all the conditional means of times of overshoots are uniformly bounded by a certain value $M(x)$ (see (3.1)). This ensues from a natural extension of the Pyke lemma. The function $M(x)$ is subadditive, as Lemma 3.2 shows, whence the estimates (2.5) and (3.5) follow. Lemma 3.3 is of a technical nature.

Let $\xi(n) = X(n) - X(n-1)$ and $X^{(l)}(n) = X(l+n) - X(l)$ for $l = 0, 1, \dots$. Set $\eta^{(l)}(x) = \min\{n\colon X^{(l)}(n) > x\}$ and $\chi^{(l)}(n) = X^{(l)}(\eta^{(l)}(x)) - x$.

LEMMA 3.1. *Under the hypotheses of Theorem* 2.1 *inequality* (2.5) *holds. Moreover, for any* $x \geqq 0$,

$$M(x) \equiv \sup_l \ \operatorname{vrai\,sup} \mathbf{E}\left\{\eta^{(l)}(x) \mid \mathcal{F}_l\right\} < \infty. \tag{3.1}$$

Proof. Taking any $0 < \varepsilon < 1$, choose $r \equiv r(\varepsilon A/m)$ to guarantee that

$$\mathbf{E}\left\{(\xi(n+1) - r)^+ \mid \mathcal{F}_n\right\} \leqq \mathbf{E}\,(\zeta - r)^+ \leqq \frac{\varepsilon A}{m}$$

a.s. for all n (here we use the notation $x^+ = \max(0, x)$).

Introducing random variables $\tilde{\xi}(n) = \min(\xi(n), r)$ we construct upon them the variables $\widetilde{X}(n) = \sum_{i=1}^n \tilde{\xi}(i)$; $\tilde{\eta}(x)$; $\widetilde{X}^{(l)}(n)$, the function $\widetilde{M}(x)$, and so on. Note the following inequalities: $M(x) \leqq \widetilde{M}(x)$ for all x and

$$\mathbf{E}\left\{\widetilde{X}^{(km+j)}(m-j) \mid \mathcal{F}_{km+j}\right\} \geqq \mathbf{E}\left\{X^{(km+j)}(m-j) \mid \mathcal{F}_{km+j}\right\} - \varepsilon A \geqq -V - \varepsilon A \equiv -\widetilde{V},$$

where V is taken from (G3). We demonstrate that

$$\widetilde{M}(x) \leqq \frac{m(x + mr + \mathbf{I}(m > 1)(V + A))}{A(1-\varepsilon)}\,. \tag{3.2}$$

For $l = km + j$ $(0 \leqq j \leqq m-1,\ k \geqq 0)$ set $Y^{(l)}(0) = 0$ and, for $n \geqq 1$, $Y^{(l)}(n) = \widetilde{X}^{(l)}(nm - j)$. Let

$$\tau^{(l)} \equiv \tau^{(l)}(x) = \min\left\{n \geqq 1\colon Y^{(l)}(n) > x\right\}.$$

Then $\widetilde{M}(x) \leqq m \sup_l \operatorname{vrai\,sup} \mathbf{E}\,\{\tau^{(l)} \mid \mathcal{F}_l\}$ and inequality (3.2) will be proved as soon as we can show that, for all l,

$$\mathbf{E}\,\{\tau^{(l)} \mid \mathcal{F}_l\} \leqq \frac{x + mr + \mathbf{I}(m > 1)(V + A)}{A(1-\varepsilon)} \quad \text{a.s.} \tag{3.3}$$

Having fixed an arbitrary l, we write, for the sake of brevity, $\tau = \tau^{(l)}$, $Y(0) = 0$, $\mathcal{F}(0) = \mathcal{F}_l$, and $Y(n) = Y^{(l)}(n)$, $\mathcal{F}(n) = \mathcal{F}_{(k+n)m}$ for $n = 1, 2, \dots$. The relations

$$\mathbf{P}\big\{Y(n+1) \leqq Y(n) + mr\big\} = 1;$$
$$\mathbf{E}\left\{Y(n+1) - Y(n) \mid \mathcal{F}_n\right\} \geqq A(1-\varepsilon) > 0 \quad \text{a.s.}$$

are true for all $n \geqq 1$. If $j = 0$, these relations hold also for $n = 0$. But if $j \geqq 1$, then $\mathbf{P}\{Y(1) \leqq (m-j)\,r\} = 1$ and $\mathbf{E}\,(Y(1) \mid \mathcal{F}(0)) \geqq -\widetilde{V}$ a.s.

Introduce a test function $L(y) = (x + mr - y)^+ \geqq 0$. Then $\tau = \min\{n \geqq 1: L(Y(n)) < mr\}$ and inequality (3.3) follows by a natural extension of the Pyke lemma (establishing a natural counterpart of (3.3) in the case of a homogeneous Markov chain $\{Y(n)\}$). To make the account complete we prove the inequality.

For any $N \geqq 1$ write

$$\tau_N = \min\{\tau, N\}; \quad F_N = \mathbf{E}\left\{\sum_1^{\tau_N} L\bigl(Y(n)\bigr) \mid \mathcal{F}_l\right\}.$$

Observe that, for any n, the event $\{\tau \geqq n\}$ enters the σ-algebra $\mathcal{F}(n-1)$. Therefore, one has (using in the following formulas the shortened notation $\mathbf{E}_l\{\cdot\}$ instead of $\mathbf{E}\{\cdot \mid \mathcal{F}_l\}$ and $\mathbf{P}_l\{\cdot\}$ instead of $\mathbf{P}\{\cdot \mid \mathcal{F}_l\}$)

$$\begin{aligned}
F_N &= \mathbf{E}_l\left(\sum_{n=1}^{N} L\bigl(Y(n)\bigr)\,\mathbf{I}(\tau \geqq n)\right) \\
&= \mathbf{E}_l L\bigl(Y(1)\bigr) + \mathbf{E}_l\left(\sum_{n=2}^{N} \mathbf{E}\left\{L\bigl(Y(n)\bigr)\mathbf{I}(\tau \geqq n) \mid \mathcal{F}(n-1)\right\}\right) \\
&= \mathbf{E}_l L\bigl(Y(1)\bigr) + \mathbf{E}_l\left(\sum_{n=2}^{N} \mathbf{I}(\tau \geqq n)\mathbf{E}\left\{L\bigl(Y(n)\bigr) \mid \mathcal{F}(n-1)\right\}\right) \\
&\leqq \mathbf{E}_l L\bigl(Y(1)\bigr) + \mathbf{E}_l\left(\sum_{n=2}^{N} \mathbf{I}(\tau \geqq n)\left[L\bigl(Y(n-1)\bigr) - A(1-\varepsilon)\right]\right) \\
&\leqq \mathbf{E}\,L\bigl(Y(1)\bigr) + \mathbf{E}_l\left(\sum_{n=2}^{N} \mathbf{I}(\tau \geqq n-1)\,L\bigl(Y(n-1)\bigr)\right) - A(1-\varepsilon)\sum_{n=2}^{N} \mathbf{P}_l\{\tau \geqq n\}.
\end{aligned}$$

In particular, if $j = 0$, then $\mathbf{E}_l Y(1) \geqq A(1-\varepsilon)$ a.s. and, for any $N = 1, 2, \dots$, we have

$$A(1-\varepsilon)\sum_{n=1}^{N} \mathbf{P}_l\{\tau \geqq n\} \leqq x + rm,$$

and if $j \geqq 1$, $\mathbf{E}_l Y(1) \geqq -\widetilde{V}$ a.s. and

$$A(1-\varepsilon)\sum_{n=1}^{N} \mathbf{P}_l\{\tau \geqq n\} \leqq x + rm + \widetilde{V} + A(1-\varepsilon) \equiv x + rm + V + A.$$

Letting N tend to infinity, we see first that $\sum_1^\infty \mathbf{P}_l\{\tau \geqq n\} = \mathbf{E}_l\tau$, and second that $j = 0$ when $m = 1$. This yields (3.3) which proves the lemma.

LEMMA 3.2. *Let $K = M(1) < \infty$. Under the hypotheses of Theorem* 2.1 *one has*

1) *for any $v \geqq 0$*

$$M(v) \leqq K(1+v); \tag{3.4}$$

2) *for any v, $t > 0$, $l = 0, 1, \dots$*

$$\mathbf{P}\bigl\{\chi^{(l)}(v) > t \mid \mathcal{F}_l\bigr\} \leqq M(v)\overline{F}(t) \quad a.s.; \tag{3.5}$$

3) *for any* v, $t > 0$, $l = 0, 1, \dots$

$$\mathbf{P}\bigl\{\chi^{(l)}(v) > t \mid \mathcal{F}_l\bigr\} \leqq K\left(\overline{F}(t) + \int_t^{t+v} \overline{F}(x)\,dx\right), \tag{3.6}$$

$$\mathbf{P}\bigl\{\chi^{(l)}(v) > t \mid \mathcal{F}_l\bigr\} \leqq c_1 \mathbf{E}\bigl\{\zeta \mathbf{I}(\zeta > t)\bigr\} \tag{3.7}$$

a.s. with c_1 *defined in* (3.9).

Proof. First, $M(v)$ is nondecreasing in $v \geqq 0$. Second, this function is subadditive: for any $u, v \geqq 0$

$$M(v+u) \leqq M(v) + M(u). \tag{3.8}$$

Indeed,

$$\begin{aligned}\eta^{(l)}(v+u) &= \eta^{(l)}(v) + \sum_{m=1}^{\infty} \int_{v+0}^{v+u} \mathbf{I}\left(\eta^{(l)}(v) = m\right)\mathbf{I}\bigl(\theta^{(l)} S(m) \in dy\bigr)\,\eta^{(l+m)}(v+u-y) \\ &\leqq \eta^{(l)}(v) + \sum_{m=1}^{\infty} \mathbf{I}\bigl(\eta^{(l)}(v) = m\bigr)\,\eta^{(l+m)}(u).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{E}\left\{\eta^{(l)}(v+u) \mid \mathcal{F}_l\right\} &\leqq \mathbf{E}\left\{\eta^{(l)}(v) \mid \mathcal{F}_l\right\} \\ &\quad + \sum_{m=1}^{\infty} \mathbf{P}\bigl\{\eta^{(l)}(v) = m \mid \mathcal{F}_l\bigr\}\operatorname{vrai\,sup}\mathbf{E}\left\{\eta^{(l+m)}(u) \mid \mathcal{F}_{l+m}\right\} \\ &\leqq M(v) + M(u).\end{aligned}$$

Consequently, for any $v > 0$, we have

$$M(v) \leqq M\bigl([v]+1\bigr) \leqq K\bigl([v]+1\bigr) \leqq K(1+v),$$

where $[v]$ is the integer part of v. We prove the second assertion (denoting $\mathbf{P}\{\cdot \mid \mathcal{F}_l\}$ by $\mathbf{P}_l\{\cdot\}$):

$$\begin{aligned}\mathbf{P}_l\bigl\{\chi^{(l)}(v) > t\bigr\} &= \sum_n \mathbf{P}_l\bigl\{\eta^{(l)}(v) = n,\ X^{(l)}(n) > t+v\bigr\} \\ &\leqq \sum_n \mathbf{P}_l\bigl\{\eta^{(l)}(v) = n,\ \xi(n+l) > t\bigr\} \\ &\leqq \sum_n \mathbf{P}_l\bigl\{\eta^{(l)}(v) \geqq n,\ \xi(n+l) > t\bigr\} \\ &\leqq \sum_n \mathbf{P}_l\bigl\{\eta^{(l)}(v) \geqq n\bigr\}\operatorname{vrai\,sup}\mathbf{P}\bigl\{\xi(n+l) > t \mid \mathcal{F}_{l+n-1}\bigr\} \\ &\leqq \overline{F}(t)\sum_n \mathbf{P}_l\bigl\{\eta^{(l)}(v) \geqq n\bigr\} \equiv \overline{F}(t)\,M(v).\end{aligned}$$

In particular, for any $v_0 \in [0,1]$ and any integer l,

$$\mathbf{P}_l\{\chi^{(l)}(v_0) > t\} \leqq K\overline{F}(t).$$

We turn to proving (3.6).

Let $v = v_0 + N$, where $0 \leqq v_0 \leqq 1$, $N \geqq 1$. Then

$$\begin{aligned}\mathbf{P}_l\bigl\{\chi^{(l)}(v) > t\bigr\} &\leqq \mathbf{P}_l\bigl\{\chi^{(l)}(v-1) > t+1\bigr\} \\ &\quad + \sum_n \int_0^1 \mathbf{P}_l\bigl\{\eta^{(l)}(v-1) = n,\ \chi^{(l)}(v-1) \in v-1+y+dy,\ \chi^{(l+n)}(1-y) > t\bigr\}\end{aligned}$$

and the second summand in the right-hand side of the last inequality is estimated from above by the expression

$$\sum_n \int_0^1 \mathbf{P}_l\big\{\eta^{(l)}(v-1)=n,\ \chi^{(l)}(v-1)\in v-1+y+dy\big\}$$
$$\times \operatorname{vrai\,sup} \mathbf{P}\big\{\chi^{(n+l)}(1-y)>t \mid \mathcal{F}_{n+l}\big\}$$
$$\leqq K\overline{F}(t)\sum_n \int_0^1 \mathbf{P}_l\big\{\eta^{(l)}(v-1)=n,\ \chi^{(l)}(v-1)\in v-1+y+dy\big\} \leqq K\overline{F}(t).$$

Proceeding by induction, we establish the estimate

$$\mathbf{P}_l\big\{\chi^{(l)}(v)>t\big\} \leqq K\sum_{k=0}^{N-1}\overline{F}(t+k).$$

The last expression does not exceed $K(\overline{F}(t)+\int_t^{t+v}\overline{F}(u)\,du)$.

Now we prove (3.7). Let $N>0$ be taken to ensure that $b(N)\equiv \mathbf{E}\{\zeta\mathbf{I}(\zeta\geqq N)\}\leqq A/(2m)$. Put $\beta=\frac{1}{2}\min\{N,1,A/(4m)\}$. Then

$$\frac{A}{m}\leqq \mathbf{E}\,\zeta = \mathbf{E}\{\zeta\mathbf{I}(\zeta<2\beta)\}+\mathbf{E}\{\zeta\mathbf{I}(2\beta\leqq\zeta<N)\}+b(N)\leqq 2\beta+N\overline{F}(2\beta)+\frac{A}{2m},$$

and, consequently, $\overline{F}(2\beta)\geqq A/(4Nm)$. Observe that, for $t\geqq\beta$

$$b(t)=t\overline{F}(t)+\int_t^\infty \overline{F}(u)\,du \geqq \beta\left(\overline{F}(t)+\int_t^\infty\overline{F}(u)\,du\right),$$

whereas for $t<\beta$

$$b(t)\geqq \int_t^\infty \overline{F}(u)\,du \geqq \int_t^{t+\beta}\overline{F}(u)\,du \geqq \beta\overline{F}(2\beta)\geqq \frac{\beta A}{4Nm}\,\overline{F}(t).$$

Therefore, for any $t,v\geqq 0$

$$\overline{F}(t)+\int_t^{t+v}\overline{F}(u)\,du \leqq b(t)\frac{1}{\beta}\,\max\left(1,\ \beta+\frac{4Nm}{A}\right),$$

and inequality (3.7) is valid when

$$\text{(3.9)}\quad c_1=\frac{K}{\beta}\,\max\left(1,\ \beta+\frac{4Nm}{A}\right);\quad \beta=\frac{1}{2}\,\min\left\{1,\ N,\ \frac{A}{4m}\right\};\quad N=b^*\left(\frac{A}{2m}\right).$$

The lemma is proved.

LEMMA 3.3. *Let $\varepsilon\in(0,1)$ be arbitrary and assume that* (G1) *holds for $m=1$. Then inequality* (3.6) *remains valid if we substitute the constant $K=M(1)$ with $K'=r(\varepsilon A)/A(1-\varepsilon)$. Here, as before, $r(t)$ is the smallest r satisfying $\mathbf{E}\,(\zeta-r)^+\leqq t$ and $r(t)\leqq b^*(t)$.*

Proof. Taking $m=1$ in the proof of Lemma 3.1, we get

$$K=M(1)\leqq \widetilde{M}(1)\leqq \frac{1+r(\varepsilon A)}{A(1-\varepsilon)}.$$

For any $z>0$ consider an auxiliary sequence $X_z(n)=X(n)/z$. It fulfills conditions (G1), (G2) with the constant $A_z=A/z$ and $F_z(t)=F(tz)$. So $r_z(t)=r(t)/z$ and, for any $z>0$,

$$\mathbf{P}\big\{\chi(v)>t\big\}\equiv \mathbf{P}\left\{\chi_z\left(\frac{v}{z}\right)>\frac{t}{z}\right\}\leqq K_z\left(\overline{F}(t)+\int_t^{t+v}\overline{F}(x)\,dx\right),$$

where

$$K_z = \frac{1 + r_z(\varepsilon A)}{A_z(1-\varepsilon)} = \frac{z + r(\varepsilon A)}{A(1-\varepsilon)}.$$

Letting z tend to zero we obtain the required assertion.

The proof of Theorem 2.1 ensues from Lemmas 3.1 and 3.2. The assertion (2.8) of Remark 2.1 follows from Lemma 3.3, and (2.9) from (3.9).

It is worth noting that we can give an alternative natural proof of Theorem 2.1 based on estimates of the renewal function (also see the remarks in subsection 2.2). Presumably, publication of the corresponding results may be expected before long.

Proof of Theorem 2.2. We begin with some remarks.

Remark 3.1. By the Strassen theorem, if $\rho(F,G) \leqq \delta$ one can define on the same probability space an r.v. ψ_1 with the distribution function F and an r.v. ψ_2 with the distribution G so that $\mathbf{P}\{|\psi_1 - \psi_2| > \delta\} \leqq \delta$.

The following fact is an immediate corollary. Let $\{X(n)\}$ be a Markov chain having increments $\xi(n) = X(n) - X(n-1)$. Write $F_{n+1} \equiv F_{n+1}(\omega)$ for the conditional expectation of $\xi(n+1)$ with respect to the σ-algebra $\mathcal{F}_n$:

$$F_{n+1}(x) = \mathbf{P}\bigl\{\xi(n+1) < x \mid \mathcal{F}_n\bigr\}.$$

Suppose now the existence of the distribution function G and of the events $A_{n+i-1} \in \mathcal{F}_{n+i-1}$ for some n and all $i = 1,\dots,l$, such that $\rho(F_{n+i}, G) \leqq \delta$ a.s. on A_{n+i-1}. Then, on the same probability space, one can define r.v.'s $\xi(1),\dots,\xi(n+l)$ and i.i.d. r.v.'s $\xi_1,\dots,\xi_l$ with the joint distribution G, so that

(a) r.v. ξ_1 is independent of $\{\xi(1),\dots,\xi(n)\}$ and, for any $i = 1,\dots,l-1$, the r.v. ξ_{i+1} is independent of $\{\xi(1),\dots,\xi(n+i),\xi_1,\dots,\xi_i\}$;

(b) the inequality

$$\mathbf{P}\bigl\{|\xi(n+i) - \xi_i| \leqq \delta \ \forall\, i = 1,\dots,l \mid \mathcal{F}_n\bigr\} \geqq 1 - l\delta - \mathbf{P}\left\{\bigcup_{i=1}^{l-1} A_{n+i} \mid \mathcal{F}_n\right\}$$

holds a.s. on the event A_n.

Remark 3.2. Consider two numerical sequences $z_1,\dots,z_l$ and $\widetilde{z}_1,\dots,\widetilde{z}_l$ such that $|z_i - \widetilde{z}_i| \leqq \delta$ for all $i = 1,\dots,l$. Set $Z_i = z_1 + \cdots + z_i$ and $\widetilde{Z}_i = \widetilde{z}_1 + \cdots + \widetilde{z}_i$. Take $x > 0$ and $t > 2l\delta$ arbitrarily and let $N_x = \min\{i\colon Z_i > x\}$, $H_x = Z_{N_x} - x$ ($\widetilde{N}_x$ and $\widetilde{H}_x$ are defined similarly). The validity of the following implication is easily seen: If $N_x \leqq l$ and $H_x > t$, then $\widetilde{N}_{x+l\delta} \leqq l$ and $\widetilde{H}_{x+l\delta} > t - l\delta$.

Now we turn directly to the proof of Theorem 2.2. Let

$$\rho(z) = \sup_{u \geqq z} \rho\Bigl(\chi_{as}(x),\ \chi_{as}(\infty)\Bigr).$$

Observe, that, for all $u \geqq z$ and $t \geqq \rho(z)$, one has

$$\mathbf{P}\bigl\{\chi_{as}(\infty) > t\bigr\} \leqq \mathbf{P}\bigl\{\chi_{as}(u) > t - \rho(z)\bigr\} + \rho(z).$$

Let at first $x = 2y + 2z$, y and z being arbitrary positive numbers. For any $N \geqq 1$,

$$\begin{aligned}\mathbf{P}\bigl\{\chi_{as}(\infty) > t\bigr\} \leqq \rho(z) &+ \mathbf{P}\bigl\{\eta(2y) < N\bigr\} + \mathbf{P}\bigl\{\chi(2y) > z\bigr\} \\ &+ \sum_{n=N}^{\infty} \int_0^z \mathbf{P}\bigl\{\eta(2y) = n;\ \chi(2y) \in dv\bigr\} \\ &\qquad \times \mathbf{P}\bigl\{\chi_{as}(x - 2y - v) > t - \rho(z)\bigr\}.\end{aligned}$$

Here $\mathbf{P}\{\chi(2y) > z\} \leqq c_1 b(z)$ and, by the Chebyshev inequality

$$\mathbf{P}\bigl\{\eta(2y) < N\bigr\} \leqq \mathbf{P}\bigl\{\eta(2y) < N\bigr\} \leqq \mathbf{P}\bigl\{\zeta_1 + \cdots + \zeta_{N-1} > 2y\bigr\} \leqq \frac{\mathbf{E}\,\zeta_1 \cdot N}{2y} \equiv \frac{bN}{2y}$$

(where $\{\zeta_n\}$ is a sequence of i.i.d. r.v.'s distributed as ζ). Further on, take $\varepsilon = \rho_{y,N}$ and integer $l \geqq 1$. According to Remarks 3.1 and 3.2, the following inequality holds a.s., for $t - \rho(z) > l\varepsilon$, $v \leqq z$, and $n \geqq N$:

$$\begin{aligned} &\mathbf{P}\bigl\{\chi_{as}(x - 2y - v) > t - \rho(z)\bigr\} \\ &\quad \leqq \mathbf{P}\Bigl\{\chi(x + l\varepsilon) > t - \rho(z) - l\varepsilon \mid \eta(2y) = n,\ \chi(2y) \in (v,\, v + dv)\Bigr\} + \mathbf{P}\bigl\{\widetilde{\eta}(2z) > l\bigr\} \\ &\qquad + l\varepsilon + \mathbf{P}\Bigl\{\min_{1 \leqq i \leqq l} X(n+i) < y \mid \eta(2y) = n,\ \chi(2y) \in (v,\, v + dv)\Bigr\}. \end{aligned}$$

By the Kolmogorov inequality the last summand in the right-hand side does not exceed $2bl/y$ and by the Chebyshev inequality $\mathbf{P}\{\widetilde{\eta}(2z) > l\} \leqq K(1 + 2z)/l$. Hence,

$$\begin{aligned} \mathbf{P}\bigl\{\chi_{as}(\infty) > t\bigr\} - \mathbf{P}\bigl\{\chi(x + l\varepsilon) > t - \rho(z) - l\varepsilon\bigr\} - \rho(z) - c_1 b(z) \\ \leqq \frac{bN}{2y} + \frac{2bl}{y} + \frac{K(1 + 2z)}{l} + l\varepsilon\ . \end{aligned}$$

Take $N \equiv N(y) = [y^{2/3}]$, $l = [\min(y, \varepsilon^{-1})]^{2/3}$, and $z \equiv z(y) = l^{1/2}$. Then (for $z \geqq 1$) the right-hand side of the inequality does not exceed

$$\frac{b}{2}\, y^{1/3} + 2by^{1/3} + \varepsilon^{1/3} + 3K\varepsilon^{1/3} \leqq \bigl(3(K + b) + 1\bigr) z^{-1}.$$

Put $Q(z) = c_1 b(z) + \rho(z) + (3(K+b)+1)z^{-1}$. Then, for $x = 2y + 2z(y)$, $t > Q(z)$, we have

$$\mathbf{P}\bigl\{\chi_{as}(\infty) > t\bigr\} \leqq \mathbf{P}\bigl\{\chi(x + l\varepsilon) > t - Q(z)\bigr\} + Q(z). \tag{3.10}$$

Now note that if we take any $x \geqq 2y + 2z(y)$ with y and $z = z(y)$ fixed, inequality (3.10) retains its validity. Indeed, it suffices to put $x = 2y' + 2z(y)$ with $y' \geqq y$ and repeat the above reasoning.

In full similarity we establish the lower bound

$$\mathbf{P}\bigl\{\chi_{as}(\infty) > t\bigr\} \geqq \mathbf{P}\bigl\{\chi(x - l\varepsilon) > t + Q(z)\bigr\} - Q(z).$$

Having observed that $l\varepsilon \leqq 1$, we establish the assertions of the theorem.

Proof of Theorem 2.3. Write

$$c(x) = \inf_{t \in [0, U(x)]} g(x, t) \quad \text{and } u(x) = \left(\frac{c(x)}{b + \Delta}\right)^{1/2}.$$

Let $\{\zeta(n)\}$ be a sequence of i.i.d. r.v.'s distributed as ζ. We mention that

$$\begin{aligned} \mathbf{P}\bigl\{\eta_g(x) < U(x)\bigr\} \leqq \mathbf{P}\bigl\{\exists n \leqq u(x)\colon X(n) > c(x)\bigr\} \\ + \mathbf{P}\Bigl\{\exists n \in \bigl(u(x),\, U(x)\bigr)\colon X(n) > (b + \Delta)\, n\Bigr\}. \end{aligned}$$

The first summand on the right-hand side does not exceed

$$\mathbf{P}\left\{\sum_{i=1}^{u(x)} \zeta(i) > c(x)\right\} \leqq \frac{bu(x)}{c(x)} \longrightarrow 0, \quad x \to \infty,$$

whereas the second one

$$\mathbf{P}\left\{\sup_{n\geqq u(x)} \frac{\zeta(1)+\cdots+\zeta(n)}{n} > b+\Delta\right\} \longrightarrow 0, \quad x\to\infty.$$

Put $U=U(x)$, $G(x,T)=\sup_{t\in[U,TU]}|g(x,t)-g(x,t+1)|$ and $v(x)=U(b+\Delta+G(x,T)\,(T-1))$. By the Chebyshev inequality for any $T>1$,

$$\begin{aligned}\mathbf{P}\{\eta_g(x)>TU\} &\leqq \mathbf{P}\Big\{\eta\big(v(x)\big)>TU\Big\}+\mathbf{P}\Big\{\eta\big(v(x)\big)\leqq U\Big\}\\ &\leqq \frac{K(1+u(x))}{TU}+o(1)\longrightarrow O\left(\frac{1}{T}\right) \quad \text{as } x\to\infty.\end{aligned}$$

Introduce auxiliary r.v. $\{\overline{X}(n)\}$, letting $\overline{X}(n)=X(n)$ for $n\leqq U$; $\overline{X}(n)=X(n)+g(x,n)-g(x,U)$ for $U<n\leqq TU$; and $\overline{X}(n)=X(n)+g(x,TU)-g(x,U)$ for $n>TU$. Set $\overline{\xi}(n)=\overline{X}(n)-\overline{X}(n-1)$, $\overline{\eta}(y)=\min\{n\colon \overline{X}(n)>y\}$, and $\overline{\chi}(y)=\overline{X}(\overline{\eta}(y))-y$. Then we have

$$\mathbf{P}\{\chi_g(x)>t\}=\mathbf{P}\Big\{\overline{\chi}\big(U(x)\big)>t\Big\}+O\left(\frac{1}{T}\right)+o(1).$$

Note the inequality valid for $\{\overline{X}(n)\}$:

$$\overline{\rho}_{y,N}\equiv \sup_{v\geqq y,m\geqq N}\rho(\overline{\xi}(v,m),\xi)\leqq G(x,T)+\rho_{y,N},$$

whence, in view of Theorem 2.6,

$$\rho\Big(\overline{\chi}\big(U(x)\big),\ \chi_{as}(\infty)\Big)=O\left(\frac{1}{T}\right)+o(1)$$

for any fixed T. Theorem 2.3 is proved.

Proof of Corollary 2.3. The first assertion is due to the fact that $\sup_{x>0}|\varphi(x)-x|$ is finite. Remark 3.2 and condition (2.16) imply the second assertion of the corollary.

Proof of Theorem 2.4. For any $\varepsilon>0$ find l so that $\mathbf{P}\{\sup_{n\geqq l}|\gamma(n)-\gamma|>\varepsilon\}\leqq\varepsilon$ and also L to ensure that $\mathbf{P}\{|\gamma|>L-\varepsilon\}\leqq\varepsilon$ and $\mathbf{P}\{|X(l)|>L\}\leqq\varepsilon$. Introduce the event $A=\{\sup_{n\geqq l}|\gamma(n)-\gamma|>\varepsilon\}\bigcup\{|\gamma|>L-\varepsilon\}\bigcup\{|X(l)|>L\}$. When $t>\varepsilon$,

$$\mathbf{P}\{\chi_{(\gamma)}(x)>t\}\leqq\mathbf{P}\{\eta_{(\gamma)}(x)\leqq l\}+\mathbf{P}\{A\}+\mathbf{P}\{\chi_{(\gamma)}(x)>t;\ \mathbf{I}(A)=0,\ \eta_{(\gamma)}(x)>l\},$$

where the last summand does not exceed

$$\begin{aligned}&\mathbf{P}\Big\{\chi^{(k)}\big(x-X(k)-\gamma(k)\big)>t-\varepsilon;\ \mathbf{I}(A)=0\Big\}\\ &\quad+\int_{-L}^{L}\int_{-L}^{L}\mathbf{P}\{\chi^{(k)}(x-u-v)>t-\varepsilon\}\mathbf{P}\{X(k)\in du;\ \gamma(k)\in dv\}.\end{aligned}$$

If $|u|\leqq L$ and $|v|\leqq L$,

$$\begin{aligned}&\mathbf{P}\{\chi^{(k)}(x-u-v)>t-\varepsilon\}-\mathbf{P}\{\chi_{as}(\infty)>t-\varepsilon\}\\ &\quad\leqq\sup_k\sup_{y\geqq x-2L}\rho(\chi^{(k)}(y),\chi_{as}(\infty))\equiv q(x-2L),\end{aligned}$$

where $q(x)\to0$ as $x\to\infty$. Consequently,

$$\mathbf{P}\{\chi_{(\gamma)}(x)>t\}\leqq\mathbf{P}\{\eta_{(\gamma)}(x)\leqq l\}+3\varepsilon+q(x-2L)+\mathbf{P}\{\chi_{as}(\infty)>t-\varepsilon\}.$$

Letting x tend to infinity we obtain the inequality

$$\limsup \mathbf{P}\{\chi_{(\gamma)}(x) > t\} \leqq 3\varepsilon + \mathbf{P}\{\chi_{as}(\infty) > t - \varepsilon\}$$

for any $\varepsilon > 0$ and $t > \varepsilon$. In full similarity one can also establish the lower estimate

$$\liminf \mathbf{P}\{\chi_{(\gamma)}(x) > t\} \geqq \mathbf{P}\{\chi_{as}(\infty) > t + \varepsilon\} - 3\varepsilon.$$

Therefore, $\limsup \rho(\chi_{(\gamma)}, \chi_{as}(\infty)) \leqq 3\varepsilon$ for any $\varepsilon > 0$ and so (2.17) is valid.

If r.v.'s $\widetilde{\gamma}$ and ζ^2 are integrable, then $\chi_{(\gamma)}(x)$ are uniformly in x integrable r.v.'s, which entails (2.18).

Proof of Theorem 2.5. Take $y \equiv y(x) = at_0(x) - x^{1/2}$. Then $\eta(y) < \eta_g(x)$ a.s. For the renewal process $\eta(y)$, $y \geqq 0$, the central limit theorem is valid: the distributions of random variables $(\eta(y) - y/a)\,(\sigma y^{1/2} a^{-3/2})^{-1}$ are weakly convergent to a standard normal distribution as $y \to \infty$. In particular, $\mathbf{P}\{\eta(y) < y/a\}$ tends to $\frac{1}{2}$ and, for $\varepsilon > 0$ small enough, one can choose numbers $0 < \delta \ll N < \infty$ to guarantee that $\mathbf{P}\{|\eta(y) - y/a| \leqq \delta y^{1/2}\} \leqq \varepsilon/4$ and $\mathbf{P}\{|\eta(y) - y/a| \geqq N y^{1/2}\} \leqq \varepsilon/4$ for all y large enough. Take $C \gg 1$ such that $\mathbf{P}\{\chi(y) > C\} \leqq \varepsilon/2$ for all y. Then, with probabilities close to $\frac{1}{2}$, the random walk will cross either a linear boundary $x + \alpha_1 t$ or a linear boundary $at_0 + \alpha_2(t - t_0)$.

It is well known that in both cases the value of overshoot (over the rectilinear boundary) converges weakly (as $x \to \infty$) to the corresponding value of overshoot over the infinitely distant barrier.

3.2. Proofs of statements of subsection 2.2.

Proof of Theorem 2.6. We begin with the claim (2.19).

Assume the value of limit in (2.19) to be positive. Then there exist $\varepsilon > 0$, $t > 0$, a sequence of distributions $\{F^{(m)}\}$, and a numerical sequence $\{x^{(m)}\}$, $x^{(m)} \to \infty$, such that

$$\left|\mathbf{P}\{\chi^{(F^{(m)})}(x^{(m)}) > t\} - \mathbf{P}\{\chi^{(F^{(m)})}(\infty) > t\}\right| \geqq 2\varepsilon. \tag{3.11}$$

Note that $\mathfrak{F}$ is a compact family: from any sequence $\{F^{(m)}\}$ belonging to $\mathfrak{F}$ one can select a subsequence $\{F^{(m_k)}\}$ weakly convergent, say, to some distribution $F \in \mathfrak{F}$. Without loss of generality, it can be assumed that already the initial sequence $\{F^{(m)}\}$ is convergent. Then r.v.'s $\chi^{(F^{(m)})}(\infty)$ converge weakly to $\chi^{(F)}(\infty)$. Since the r.v. $\chi^{(F)}(\infty)$ has continuous distribution, (3.11) implies that $|\mathbf{P}\{\chi^{(F^{(m)})}(x^{(m)}) > t\} - \mathbf{P}\{\chi^{(F)}(\infty) > t\}| \geqq \varepsilon$ for all m large enough. We will not lose in generality by assuming that this inequality holds for all m.

For any fixed $0 < \delta < t$, choose $y > \delta$ so that $\rho(\chi^{(F)}(z), \chi^{(F)}(\infty)) \leqq \delta$ for all $z \geqq y - \delta$.

By Corollary 2.1,

$$\sup_{F \in \mathfrak{F}} \sup_{x} \mathbf{P}\{\chi^{(F)}(x) > t\} \leqq C_1 B(t).$$

Hence we can choose a constant $C > 0$ so that $\mathbf{P}\{\chi^{(F)}(x) > C\} \leqq \delta$ for all x and all $F \in \mathfrak{F}$.

Put $l = [K(y + C)/\delta] + 1$, K being the constant defined in Lemma 3.2. For any function $F \in \mathfrak{F}$,

$$\mathbf{P}\{\eta^{(F)}(y + C) > l\} \leqq \frac{K(y + C)}{l} \leqq \delta.$$

Finally, take $m_0 \equiv m_0(\delta)$ such that $x_{m_0} > y + C + \delta$ and $\rho(F^{(m)}, F) \leqq \delta/l$ for any $m \geqq m_0$. In this case

$$\mathbf{P}\big\{\chi^{(F^{(m)})}(x^{(m)}) > t\big\}$$
$$= \int_0^\infty \mathbf{P}\Big\{\chi^{(F^{(m)})}(x^{(m)}) > t,\ \chi^{(F^{(m)})}(x^{(m)} - y - C) \in (u,\, u + du)\Big\} = O(\delta) + \int_0^C .$$

From Remarks 3.1 and 3.2 one deduces that, uniformly in u, $0 < u \leqq C$,

$$\begin{aligned}
&\mathbf{P}\big\{\chi^{(F^{(m)})}(y + C - u) > t\big\}\\
&\quad = \mathbf{P}\big\{\chi^{(F^{(m)})}(y + C - u) > t, \eta^{(F^{(m)})}(y + C - u) \leqq l\big\} + O(\delta)\\
&\quad \leqq \mathbf{P}\big\{\chi^{(F)}(y + \delta + C - u) > t - \delta\big\} + O(2\delta)\\
&\quad \leqq \mathbf{P}\big\{\chi^{(F)}(\infty) > t\big\} + O(3\delta).
\end{aligned}$$

In the same manner it can be shown that

$$\mathbf{P}\big\{\chi^{(F^{(m)})}(y + C - u) > t\big\} \geqq \mathbf{P}\big\{\chi^{(F)}(\infty) > t\big\} - O(3\delta).$$

Thus,

$$\Big|\mathbf{P}\big\{\chi^{(F^{(m)})}(x^{(m)}) > t\big\} - \mathbf{P}\big\{\chi^{(F)}(\infty) > t\big\}\Big| \leqq 5\delta$$

for any $\delta > 0$ and $m \geqq m_0(\delta)$, which contradicts assumption (3.11). So we have proved assertion (2.19).

Passing to the proof of relation (2.21) we need some auxiliary statements.

Let $S^{(F)}(n)$ be a sequence of sums of i.i.d. r.v.'s with distribution F and

$$h^{(F)}(y,\, v) = \mathbf{E}\left(\sum_1^{\eta^{(F)}(v)} \mathbf{I}\left(S^{(F)}(n) \leqq y\right)\right). \tag{3.12}$$

Observe that $h^{(F)}(y, v) \leqq \mathbf{E}\,\eta^{(F)}(v)$ for any $v \geqq 0$ and y.

LEMMA 3.4. *For any function $F \in \mathfrak{F}$ and numbers $v \geqq 0$ and y we have*

$$H^{(F)}(y,\, v) \leqq h^{(F)}(y + v,\, v) \leqq \mathbf{E}\,\eta^{(F)}(v).$$

In particular, $H^{(F)}(y, v) \leqq K(1 + v)$, where K is the same for all F constant defined by way of (2.8).

Proof. Having fixed F, set $H = H^{(F)}$, $h = h^{(F)}$, $\eta = \eta^{(F)}$, $\chi = \chi^{(F)}$. For $N = 1, 2, \ldots$ we write

$$H^N(y) = \mathbf{E}\left(\sum_{n=1}^N \mathbf{I}(S(n) \leqq y)\right) \text{ and } h^N(y, v) = \mathbf{E}\left(\sum_1^{\min(N,\eta(v))} \mathbf{I}(S(n) \leqq y)\right),$$

where $S(n)$ is a sequence of sums of i.i.d. r.v.'s with distribution F. Then, for any fixed $v \geqq 0$ and y, the sequence of differences $H^N(y + v) - H^N(y)$ is nondecreasing in N and $H(y, v) = \lim_{N\to\infty}(H^N(y + v) - H^N(y))$, whereas the sequence $h^N(y, v)$

is monotonically increasing in N and tends to $h(y,v)$ as $N \to \infty$. The following relations hold for any N:

$$\begin{aligned} H^N(y) &= \int_{-\infty}^{y} dH^N(x) = \int_{-\infty}^{y} d_x h^N(x,\, v) \\ &\quad + \sum_{k=1}^{N-1} \int_{z=0}^{\infty} \int_{x=-\infty}^{y-v-z} \mathbf{P}\bigl\{\chi(v) \in dz,\ \eta(v) = k\bigr\}\, dH^{N-k}(x) \\ &\leqq h^N(y,\, v) + \int_0^{\infty} \int_{-\infty}^{y-v-z} \mathbf{P}\bigl\{\chi(v) \in dz,\ \eta(v) < N\bigr\}\, dH^N(x) \\ &\leqq h(y,\, v) + H^N(y-v). \end{aligned}$$

Therefore, $H^N(y) - H^N(y-v) \leqq h(y,v)$ for all N and, consequently, $H(y,v) \leqq h(y+v,v) \leqq \mathbf{E}\,\eta(v) \leqq K(1+v)$ for any $v \geqq 0$ and y.

LEMMA 3.5. *The families of random variables* $\{\chi^{(F)}(v)\}_{F\in\mathfrak{F}}$ *and* $\{\eta^{(F)}(v)\}_{F\in\mathfrak{F}}$ *are uniformly in* $F \in \mathfrak{F}$ *integrable for any fixed* $v \geqq 0$, *provided that conditions* (i), (ii)$'$, *and* (iii) *hold.*

Proof. For any fixed function F, from (3.4) and (3.5) we get the following estimate:

$$\mathbf{P}\bigl\{\chi(v) > x\bigr\} \leqq \mathbf{P}\{\xi_1 > x\}\mathbf{E}\,\eta(v) \leqq K(1+v)\,\mathbf{P}\{\xi_1 > x\}.$$

Since the constant K is the same for all F, the uniform integrability of the family $\{\chi^{(F)}(v)\}$ follows.

The task is now to show the same for the second family. As in the proof of the first part of the theorem, it will be enough to demonstrate that a sequence $\{\eta^{(F^{(m)})}\}$ is uniformly integrable, whenever $F^{(m)} \in \mathfrak{F}$ weakly converge to F and $a(F^{(m)}) \to a(F)$.

We point out that

$$\mathbf{P}\Bigl\{S^{(F)}(n) \neq y \ \ \forall\, n \leqq \eta^{(F)}(y)\Bigr\} = 1 \tag{3.13}$$

for all y, with possible countable exceptions. Choose $y \geqq v$ rendering the validity of (3.13). Then $\chi^{(F^{(m)})}(y)$ are weakly convergent to $\chi^{(F)}(y)$ and, respectively, $\eta^{(F^{(m)})}(y)$ to $\eta^{(F)}(y)$. Indeed, for any fixed n one has

$$\begin{aligned} \mathbf{P}\bigl\{\eta^{(F^{(m)})}(y) > n\bigr\} \ &= \ \mathbf{P}\bigl\{S^{(F^{(m)})}(k) \leqq y \ \ \forall\, k \leqq n\bigr\} \\ &\longrightarrow \mathbf{P}\bigl\{S^{(F)}(k) \leqq y \ \ \forall\, k \leqq n\bigr\} = \mathbf{P}\bigl\{\eta^{(F)}(y) > n\bigr\} \end{aligned}$$

and, for any $v \geqq 0$,

$$\mathbf{P}\bigl\{\chi^{(F^{(m)})}(y) > v\bigr\} = \sum_{n=1}^{\infty} \mathbf{P}\bigl\{S^{(F^{(m)})}(n) > y+v,\ S^{(F^{(m)})}(k) \leqq y \ \ \forall\, k < n\bigr\}.$$

Each term of the last series converges to the required limit, whereas the “tails” of the series are bounded uniformly in m.

Note further that, by virtue of uniform integrability, $\mathbf{E}\,\chi^{(F^{(m)})}(y) \longrightarrow \mathbf{E}\,\chi^{(F)}(y)$. From the representation $\chi(y) = S(\eta(y)) - y$ and the Wald identity we infer the convergence

$$\mathbf{E}\,\eta^{(F^{(m)})}(y) = \frac{\mathbf{E}\,\chi^{(F^{(m)})}(y) + y}{a(F^{(m)})} \longrightarrow \frac{\mathbf{E}\,\chi^{(F)}(y) + y}{a(F)} = \mathbf{E}\,\eta^{(F)}(y).$$

It is well known that if a sequence of nonnegative r.v.'s converges weakly and also if the respective means converge to a finite mean of the limiting r.v., then the sequence of r.v.'s is uniformly integrable. This establishes the uniform integrability of the family $\{\eta^{(F^{(m)})}(y)\}$. Since $0 \leqq \eta^{(F^{(m)})}(v) \leqq \eta^{(F^{(m)})}(y)$ a.s., the family $\{\eta^{(F^{(m)})}(v)\}$ is also uniformly integrable. The lemma is proved.

COROLLARY 3.1. *For any fixed $y > 0$, the family of r.v.'s $\{\eta^{(F)}(u+y) - \eta^{(F)}(u)\}$ is integrable uniformly in $F \in \mathfrak{F}$ and $u \geqq 0$.*

In fact, since $S^{(F)}(\eta^{(F)}(u)) = \chi^{(F)}(u) + u \geqq u$, $\eta^{(F)}(u)$ being a Markov time, one has

$$\begin{aligned} 0 &\leqq \eta^{(F)}(u+y) - \eta^{(F)}(u) = \min\Big\{n \geqq 0\colon S^{(F)}\big(n + \eta^{(F)}(u)\big) > u + y\Big\} \\ &= \min\big\{n \geqq 0\colon \chi^{(F)}(u) + \widetilde{S}^{(F)}(n) > y\big\} \leqq \min\big\{n \geqq 0\colon \widetilde{S}^{(F)}(n) > y\big\} \stackrel{\mathrm{d}}{=} \eta^{(F)}(y), \end{aligned}$$

where

$$\widetilde{S}^{(F)}(0) = 0, \qquad \widetilde{S}^{(F)}(n) = \sum_{i=1}^{n} \tilde{\xi}_i^{(F)}, \qquad \tilde{\xi}_i^{(F)} = \xi^{(F)}(i + \eta^{(F)}(u)), \tag{3.14}$$

and $\{\tilde{\xi}_i^{(F)}\}$ form a sequence of independent r.v.'s distributed as $\xi_1^{(F)}$ and independent of $\chi^{(F)}(u)$. Additionally, in view of the uniform in F integrability of the family of r.v.'s $\{\eta^{(F)}(y)\}$, the same is true of $\{\eta^{(F)}(u+y) - \eta^{(F)}(u)\}$. The corollary is proved.

Because of the relation

$$\psi^{(F)}(u, y, v) \equiv \sum_{n=\eta^{(F)}(u)}^{\eta^{(F)}(u+y)-1} \mathbf{I}\big(S^{(F)}(n) \in (u,\, u+v]\big) \leqq \eta^{(F)}(u+y) - \eta^{(F)}(u),$$

the last statement implies uniform, in $F \in \mathfrak{F}$ and $u \geqq 0$, integrability of the family of r.v.'s $\{\psi^{(F)}(u, y, v)\}$ for any fixed y, $v \geqq 0$.

COROLLARY 3.2. *For any fixed $v \geqq 0$ uniformly in $\mathcal{F} \in \mathfrak{F}$*

$$h^{(F)}(y,\, v) \to 0$$

as $y \to -\infty$.

Indeed, for any $\varepsilon > 0$, we can find $N \equiv N(\varepsilon)$ such that $\mathbf{E}\,(\eta^{(F)}(v)\mathbf{I}(\eta^{(F)}(v) \geqq N)) \leqq \varepsilon$ with any F. Then

$$h^{(F)}(y, v) \leqq \varepsilon + N\mathbf{P}\left\{-\sum_{k=1}^{N} \big|\xi^{(F)}(k)\big| \leqq y\right\} \leqq \varepsilon + N^2 \frac{B(0)}{|y|} \longrightarrow \varepsilon$$

as $y \to -\infty$. The corollary is proved.

For any $y \geqq v$ the following representation is valid:

$$\begin{aligned} H^{(F)}(u,\, v) &= \mathbf{E}\left(\sum_{n=\eta^{(F)}(u)}^{\infty} \mathbf{I}\big(S^{(F)}(n) \in (u,\, u+v]\big)\right) \\ &= \mathbf{E}\big\{\psi^{(F)}(u,\, y,\, v)\big\} + \int_0^\infty \mathbf{P}\big\{\chi^{(F)}(u+y) \in dz\big\}\, H^{(F)}(-y-z,\, v) \\ &\equiv E_1^{(F)}(u,\, y,\, v) + E_2^{(F)}(u,\, y,\, v), \end{aligned}$$

where, for v fixed, $0 \leqq E_2^{(F)}(u,y,v) \leqq \sup_{x \geqq y} H^{(F)}(-x,v) \longrightarrow 0$ uniformly in $F \in \mathfrak{F}$ as $y \to \infty$. In quite the same manner, having written

$$\psi^{(F)}(\infty,\, y,\, v) = \sum_{n=0}^{\widetilde{\eta}^{(F)}(y)-1} \mathbf{I}\left(\chi^{(F)}(\infty) + S^{(F)}(n) \in (0,\, v]\right),$$

with $\chi^{(F)}(\infty)$ independent of $\{\xi_n^{(F)}\}$ and $\widetilde{\eta}^{(F)}(y) = \min\{n \geqq 0\colon \chi^{(F)}(\infty) + S^{(F)}(n) > y\}$, we establish that

$$\frac{v}{a(F)} = \lim_{u\to\infty} H^{(F)}(u,\, v) \equiv H^{(F)}(\infty,\, v) = \mathbf{E}\,\psi^{(F)}(\infty,\, y,\, v) + E_2^{(F)}(\infty,\, y,\, v),$$

where $0 \leqq E_2^{(F)}(\infty,y,v) \leqq \sup_{x\geqq y} H^{(F)}(-x,v)$.

We pass directly to the proof of (2.21). It suffices to demonstrate that, given a sequence of distribution functions $\{F^{(m)}\}$ and a numerical sequence $u^{(m)} \to \infty$ such that $\{F^{(m)}\}$ converges weakly to $F \equiv F^{(\infty)}$, the limit $\lim_{m\to\infty} H^{(F^{(m)})}(u^{(m)}, v)$ exists and coincides with $v/a(F)$.

Take an arbitrary $\varepsilon > 0$ and find $y = y(\varepsilon)$ so that $H^{(F)}(-x,v) \leqq \varepsilon$ for all $x \geqq y$ and $F \in \mathfrak{F}$. Then we get

$$\left| H^{(F^{(m)})}(u^{(m)},\, v) - \frac{v}{a(F)} \right| \leqq \left| \mathbf{E}\,\psi^{(F^{(m)})}(u^{(m)},\, y,\, v) - \mathbf{E}\,\psi^{(F)}(\infty,\, y,\, v) \right| + 2\varepsilon.$$

As was already proved in the first part of the theorem, r.v.'s $\chi^{(F^{(m)})}(u^{(m)})$ weakly converge to the r.v. $\chi^{(F)}(\infty)$ having continuous distribution. Therefore, for each $n = 1, 2, \ldots$ we have weak convergence of r.v.'s $\mathbf{I}(\chi^{(F^{(m)})})(u^{(m)} + \widetilde{S}^{(F^{(m)})}(n)) \in (0,v])$ to $\mathbf{I}(\chi^{(F)}(\infty) + S^{(F)}(n) \in (0,v])$; convergence of r.v.'s $(\eta^{(F^{(m)})})(u^{(m)} + y) - \eta^{(F^{(m)})}(u^{(m)})$ to $\widetilde{\eta}^{(F)}(y)$; and convergence of r.v.'s $\psi^{(F^{(m)})}(u^{(m)}, y, v)$ to $\psi^{(F)}(\infty, y, v)$.

Due to the uniform integrability of the last r.v.'s their means $\mathbf{E}\,\psi^{(F^{(m)})}(u^{(m)}, y, v)$ converge to $\mathbf{E}\,\psi^{(F)}(\infty, y, v)$. In view of the arbitrariness of $\varepsilon > 0$, the assertion (2.21) holds. The theorem is proved.

The proofs of Theorems 2.7 and 2.8 are different from that of Theorem 2.6 only in minor modifications, so we omit them.

3.3. Proof of Theorem 2.9.

LEMMA 3.6. *For $\eta \equiv \eta(x)$ one has*

$$\mathbf{E}\,\eta = \frac{x}{a}\,, \qquad \mathbf{E}\,\eta < \frac{x}{a} + u(x), \qquad \mathbf{E}\,(\eta - \mathbf{E}\,\eta)^2 < cx.$$

Proof. The first two claims are obvious. The last one is known for the linear boundaries $h(t) = \varepsilon t$ and $\theta(n) \equiv 0$ (when $< cx$ is changed for $\approx (\sigma^2/a^3)\, x$).

If $\theta(n) \to \theta$, $\mathbf{E}\,(\theta(n))^2 < c < \infty$, the standard argument on convergence of $\eta(x)$ (for linear boundaries) to the normal distribution together with two moments does not need any essential modification. The same refers to the uniformity of this convergence as $\varepsilon \to 0$.

So we will assume that

$$\mathbf{E}\,(\eta - \mathbf{E}\,\eta)^2 < cx \tag{3.15}$$

for $x \gg 1$, provided that $\theta(n) \to \theta$ and $h(t) = \varepsilon t$ is any linear boundary. For arbitrary boundaries h we have

$$\begin{aligned}\mathbf{E}\,(\eta - \mathbf{E}\,\eta)^2 &= \mathbf{E}\Big((\eta - \mathbf{E}\,\eta)^2 \mathbf{I}\,(\eta > \mathbf{E}\,\eta)\Big) + \mathbf{E}\,\big((\eta - \mathbf{E}\,\eta)^2 \mathbf{I}\,(\eta \leqq \mathbf{E}\,\eta)\big)\\ &\leqq \mathbf{E}\,\big((\eta_{\rm tangent} - \mathbf{E}\,\eta)^2 \mathbf{I}\,(\eta > \mathbf{E}\,\eta)\big) + \mathbf{E}\,\big((\eta_{\rm chord} - \mathbf{E}\,\eta)^2 \mathbf{I}\,(\eta \leqq \mathbf{E}\,\eta)\big),\end{aligned}$$

where $\eta_{\rm tangent}$ and $\eta_{\rm chord}$ are the respective times of crossing the linear boundaries $g_{\rm tangent}(x,t) = x + h(E) + (t-E)h'(E)$ and $g_{\rm chord}(x,t) = x + th(E)/E$. However, the values $\mathbf{E}\,\eta_{\rm tangent}$ and $\mathbf{E}\,\eta_{\rm chord}$ are known to equal $\mathbf{E}\,\eta + C_1$ and $\mathbf{E}\,\eta + C_2$, respectively (for some constants C_1, C_2). Therefore,

$$\mathbf{E}\,(\eta - \mathbf{E}\,\eta)^2 \leqq \mathbf{E}\,(\eta_{\rm tangent} - \mathbf{E}\,\eta_{\rm tangent} - C_1)^2 + \mathbf{E}\,(\eta_{\rm chord} - \mathbf{E}\,\eta_{\rm chord} - C_2)^2 < cx$$

for $x \gg 1$ in view of (3.15).

Now we can pass to proving the theorem. Note that the sequence $S(n)$ forms a martingale. Hence $\mathbf{E}\,S(\eta) = a\mathbf{E}\,\eta \equiv aE$. Because of $S(\eta) = X(\eta) - \theta + o(1) = x + h(\eta) + \chi - \theta + o(1)$, we have $aE = x + \mathbf{E}\,h(\eta) + d + o(1)$, and, on account of condition (g3),

$$\mathbf{E}\,h(\eta) = \mathbf{E}\,h\big(E + (\eta - E)\big) = h(E) + \frac{\mathbf{E}\,(\eta - E)^2}{2E^2}\,h(E)\,\mathbf{E}\,c(E, \eta - E),$$

where the first factor in the last summand has order $(x/x^2)\,x^{\alpha} = x^{\alpha-1} \longrightarrow 0$ and the second factor is bounded. Therefore $\mathbf{E}\,g(\eta) = g(E) + o(1)$ and we obtain an equation for E: $aE = x + h(E) + d + o(1)$. The continuous dependence of the solution in $o(1)$ is obvious. The theorem is proved.

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