COMPARISON OF SERVICING STRATEGIES IN MULTICHANNEL QUEUEING SYSTEMS

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1. Introduction. In [1] (cf. also [2-4]) a class of m-channel queueing systems was considered which is defined by two sequences of nonnegative random variables $\{\tau_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=1}^{\infty}$ [where τ_1 is the moment of arrival of the first call; τ_i $(i \geq 2)$ the time between arrival of the $(i-1)$-th and i-th calls; s_i $(i \geq 1)$ is the time required to service the i-th call], by the vector $\overline{w}_0 = (w_{0,1}; w_{0,2}; \ldots w_{0,m})$ ($w_{0,j}$ the nonnegative instant after which the j-th channel can start to service calls with indices 1, 2, . . .), and a class of service strategies. A strategy R was taken to be a sequence $R = \{R_i\}_{i=1}^{\infty}$ of random variables, where R_i is the number of the channel at which the i-th call queues up for servicing, calls being serviced in every channel in their order of arrival.

It was assumed that

1) the distributions $s_1, s_2, \ldots, s_n, \ldots$ are conditionally commutative, i.e., if $n \in N$, $\{B_1, B_2, \ldots, B_n\}$ are arbitrary Borel sets, and $\{C_1, C_2, \ldots, C_n\}$ is any permutation of the B's, then $P\{s_1 \in C_1; \ldots; s_n \in C_n/\overline{w}_0; \{\tau_i\}_{i=1}^{\infty}\} = P\{s_1 \in B_1; \ldots; s_n \in B_n/\overline{w}_0; \{\tau_i\}_{i=1}^{\infty}\}$ a.s.;

2) for $i \geq 1$ R_i does not depend on the set of random variables $\{s_i; s_{i+1}; \ldots; s_n; \ldots\}$.

Under these conditions, it was shown that

3) the FCFS (first comes first serve) strategy minimizes the distribution $\max\limits_{1 \leq k \leq m} w_{n,k}$ for every $n \in N$ (i.e., $P\left\{\max\limits_{1 \leq k \leq m} w_{n,k} < x\right\}$ is maximal for every x), where $w_{n,k}$ $(1 \leq k \leq m)$ is the time from arrival of the n-th call for servicing to the time when servicing of the first n calls is complete in the k-th channel;

4) a class of functionals whose distributions are minimized by FCFS was described.

However, we remark that when taking such an approach to determining strategies, certain strategies (which in particular are used in practice) are not contained in the class as defined above. For example, for the FCFS strategy when two calls arrive at the same channel, the call which arrived first may be serviced after the other call; this is not possible for the class of strategies introduced in [1].

In this note, we obtain results analogous to 3)-4) for another class of strategies: these strategies determine the order in which calls are serviced, and in any given channel this order can differ from the order in which the calls arrive.

2. Definitions and Statement of Results. In order to keep the exposition simple, in what follows we will use conditions which are stronger than 1):

5) $\{s_i\}_{i=1}^{\infty}$ are jointly independent and identically distributed;

6) $\{s_i\}_{i=1}^{\infty}$ does not depend on $\overline{w}_0$ or $\{\tau_i\}_{i=1}^{\infty}$.

Let $\langle \Omega, F, P \rangle$ be a probability space on which all the random variables considered are defined and measurable with respect to the σ-algebra F with probability measure P; we write $N = \{1, 2, \ldots, n, \ldots\}$ and for $j = 1, 2, \ldots$ $x_j = \sum\limits_{i=1}^{j} \tau_i$ is the time of arrival of the j-th call. The calls are numbered in order of arrival.

Let $\{\nu_i\}_{i=1}^{\infty}$ be a random permutation of the set $\{1, 2, \ldots, n, \ldots\}$, i.e., $\{\nu_i\}_{i=1}^{\infty}$ is a sequence of random variables such that for all $i, j \in N$, $i \neq j$

$$P\left\{\bigcup_{k=1}^{\infty} \{\nu_k = j\}\right\} = P\left\{\bigcup_{k=1}^{\infty} \{\nu_i = k\}\right\} = P\{\nu_i \neq \nu_j\} = 1. \tag{1}$$

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A strategy R is a sequence

$$R=\{\nu_i\}_{i=1}^{\infty}. \tag{2}$$

The class of all strategies is denoted by T.

A queueing system in which for $j \in N$ the j-th call is serviced for a time s_j is called a system of type 1 and denoted by $\langle\{s_i\}, 1\rangle$.

A strategy $R \in T$ in systems of type 1 defines the order of arrival of calls for servicing in the following way:

For $1 \le l \le m$ let $u_{0,l} = w_{0,l}$ and $t_1 = \min_{1\le l\le m} u_{0,l}$ be the time after which calls with numbers 1, 2, . . . , n, . . . can be serviced in the system. The first (j-th) call to be served is the one for which $\nu_j = 1$. This call begins to be serviced at time $\alpha_1 = \max\{t_1; x_j\}$ in the channel with number $l_1 = \min\{l: 1 \le l \le m;\ u_{0,l} \le \alpha_1\}$ (i.e., the channel having minimal number among the free channels), and servicing continues for a time s_j.

For $j \in N$, on the set $\{\nu_j = 1\}$ we put $u_{1,l} = u_{0,l}$ for $l \ne l_1$, $1 \le l \le m$ and $u_{1,l_1} = \alpha_1 + s_j$.

Assume that for $k \in N$ the random variables α_k have been defined, where α_k is the time when the k-th call to be serviced starts to be serviced, and let $u_{k,l}$ be the time when servicing in the l-th channel of all calls j such that $\nu_j \le k$, $t_{k+1} = \min_{1\le l\le m} u_{k,l}$ is complete.

On the set $\{\nu_j = k+1\}$ for $j \in N$, until time $\alpha_{k+1} = \max\{\alpha_k; t_{k+1}; x_j\}$ calls with numbers r such that $\nu_r \ge k+1$ are not serviced, and at time α_{k+1}, servicing of the j-th call commences in the channel with number $l_{k+1} = \min\{l: 1 \le l \le m;\ u_{k,l} \le \alpha_{k+1}\}$ and continues for a time s_j.

For $j \in N$ and on the set $\{\nu_j = k+1\}$, we put $u_{k+1,l} = u_{k,l}$ for $l \ne l_{k+1}$, $1 \le l \le m$ and $u_{k+1,l_{k+1}} = \alpha_{k+1} + s_j$.

The order of arrival of calls for servicing in a system of type 1 is thereby described.

For $A \in F$, let $I\{A\}$ be the characteristic function of the set A, and for $j \in N$ let $y_j = \sum_{k=1}^{\infty} \alpha_k I\{\nu_j = k\} = \alpha_{\nu_j}$ be the time at which servicing of the j-th call commences, $z_j + y_j + s_j$ the time when servicing of the j-th call is complete. We remark that for $i, j \in N$ and on the set $\{\nu_j < \nu_i\}$, we have the inequality $y_j \le y_i$.

Consider the class $T_1 = T_1(\{s_i\}) \subset T$ of strategies defined as follows: $R = \{\nu_i\}_{i=1}^{\infty} \in T_1$ if for every $l \ge 1$ and every set of numbers $\{j_1, \ldots, j_l\}$, $j_k \in N$ with $1 \le k \le l$ we have the equality

$$P\{\nu_{j_1} = 1; \ldots; \nu_{j_l} = l/\overline{w}_0;\ \{\tau_i\}_{i=1}^{\infty}; \{s_i\}_{i=j}^{\infty}\} = P\{\nu_{j_1} = 1; \ldots; \nu_{j_l} = l/\overline{w}_0;\ \{\tau_i\}_{i=1}^{\infty}; \{s_{j_i}\}_{i=1}^{l-1}\}\ \text{a.s.} \tag{3}$$

(i.e., queueing does not depend on the servicing time of calls which have not yet arrived).

We remark that the FCFS $R^{(0)} = \{\nu_i^{(0)}\}_{i=1}^{\infty}$ and LCFS $R^* = \{\nu_i^*\}_{i=1}^{\infty}$ strategies belong to the class T_1 and are defined as follows: for $j, k \in N$, $\nu_j^{(0)} = j$ a.s. and

$$\{\nu_j^* = 1\} = \{x_j \le t_1;\ x_{j+1} > t_1\};$$

$$\{\nu_j^* = k+1\} = \bigcup_{l=0}^{\infty} \{r(t_{k+1}) = j + l;\ \nu_j > k;\ \nu_{j+s} \le k \ \text{for}\ \ 1 \le s \le l\},$$

where $r(t) = \max\{j: x_j \le t\}$ for $t \ge 0$.

Consider a system of type 1 and let $n \in N$, $1 \le k \le m$, $R \in T$, $w_{n,k} = w_{n,k}(R)$ be the time from arrival of the n-th call for servicing until completion of servicing of the first n calls in the k-th channel using strategy R; $w_{n,k}^{(0)} = w_{n,k}(R^{(0)})$.

<u>THEOREM 1.</u> In a type 1 system, we have for all $R \in T_1$, $a \ge 0$,

$$n \in N\ \ P\left\{\max_{1\le k\le m} w_{n,k}^{(0)} < a\right\} \ge P\left\{\max_{1\le k\le m} w_{n,k} < a\right\}. \tag{4}$$

<u>3. Auxiliary Results.</u> We consider a somewhat different queueing system $\langle\{s_i\}, 2\rangle$ of type 2. When $R \in T$, we have for every $j, k \in N$ and on the set $\{\nu_j = k\}$ that the j-th call is serviced for a time s_k, and not s_j as in systems of type 1 (in other words, in type 1 systems the j-th call to arrive is serviced for time s_j, while in type 2 systems it is the k-th call to be serviced (counting in order) and the time of servicing is s_k). The procedure by which calls arrive for servicing in type 2 systems is the same as for type 1 systems, except for the following

change: let $j,\ k \in N$; on the set $\{\nu_j = k\}$ (the j-th call to arrive is the k-th call to be serviced) we have the equality $u_{k,l_k} = \alpha_k + s_j$ in type 1 systems, while for type 2 systems we have $u_{k,l_k} = \alpha_k + s_k$. Retaining for type 2 systems the notation used for type 1 systems, we have for $R \in T$

$$y_j = \sum_{k=1}^{\infty} \alpha_k I\{\nu_j = k\} = \alpha_{\nu_j};$$

$$z_j = \sum_{k=1}^{\infty} s_k I\{\nu_j = k\} + y_j = \alpha_{\nu_j} + s_{\nu_j}.$$

We define a class of strategies $T_2 = T_2(\{s_i\}) \subset T$ as follows: $R \in T_2$ if for every $l \geq 1$ and all l-tuples of numbers $\{j_1, \ldots, j_l\}$ with $j_k \in N$ for $1 \leq k \leq l$, we have the equality

$$P\{\nu_{j_1} = 1, \ldots, \nu_{j_l} = l/\overline{w}_0;\ \{\tau_i\}_{i=1}^{\infty};\ \{s_i\}_{i=1}^{\infty}\} = P\{\nu_{j_1} = 1, \ldots, \nu_{j_l} = l/\overline{w}_0;\ \{\tau_i\}_{i=1}^{\infty};\ \{s_i\}_{i=1}^{l-1}\} \text{ a.s.} \tag{5}$$

(i.e., the sequence in which calls are serviced does not depend on the time for servicing of calls which have not yet arrived).

For $R = \{\nu_i\}_{i=1}^{\infty} \in T$, let $R^{-1} = \{\mu_i\}_{i=1}^{\infty} \in T$ be the strategy such that for all $j,\ k \in N$ the equality $\mu_k = j$ holds on the set $\{\nu_j = k\}$. For $k \in N$ we put

$$s_k' = s_k'(R) = \sum_{j=1}^{\infty} I\{\nu_j = k\}\, s_j = s_{\mu_k};$$

$$s_k'' = s_k''(R) = \sum_{j=1}^{\infty} I\{\nu_k = j\}\, s_j = s_{\nu_k}.$$

Remark 1. In what follows, all the proofs can be given (without loss of generality) under the assumption that $w_{0,1}, \ldots, w_{0,m}, \{\tau_i\}_{i=1}^{\infty}$ are arbitrary fixed numbers.

LEMMA 1. 1) Let $R \in T_1(\{s_i\})$. Then the $\{s_k'\}_{k=1}^{\infty}$ are jointly independent and have the same distribution as s_1.

2) Let $R \in T_2(\{s_i\})$. Then the $\{s_k''\}_{k=1}^{\infty}$ are jointly independent and have the same distribution as s_1.

Proof. We prove 1), assertion 2) being proved entirely analogously.

We verify that for $k \in N$ s_k' has the same distribution as s_1. Indeed, for $a \geq 0$

$$P\{s_k' < a\} = \sum_{j=1}^{\infty} P(\nu_j = k;\ s_j < a\} = \sum_{j=1}^{\infty} P(\nu_j = k\}\ P\{s_j < a\} = P\{s_1 < a\} \cdot \sum_{j=1}^{\infty} P\{\nu_j = k\} = P\{s_1 < a\}.$$

We check that the $\{s_k'\}_{k=1}^{\infty}$ are independent. It suffices to show that for all $n \in N$ and all natural numbers $k_1, \ldots, k_n$, we have

$$P\{s_{k_1}' < a_1;\ \ldots;\ s_{k_n}' < a_n\} = P\{s_{k_1}' < a_1\} \cdot \ldots \cdot P\{s_{k_n}' < a_n\}$$

for all $(a_1, \ldots, a_n) \in F_+^n$. In order to simplify the exposition, we give the proof for n = 2:

$$P\{s_{k_1}' < a_1;\ s_{k_2}' < a_2\} = \sum_{j_1 \neq j_2} P\{s_{k_1}' < a_1;$$

$$\nu_{k_1} = j_1;\ s_{k_2}' < a;$$

$$\nu_{k_2} = j_2\} = \Sigma_1 + \Sigma_2;$$

$$\Sigma_1 = \sum_{j_1 < j_2} P\{s_{j_1} < a_1;\ \nu_{k_1} = j_1;\ s_{j_2} < a_2;\ \nu_{k_2} = j_2\} =$$

$$= \sum_{j_1 < j_2} P\{s_{j_1} < a_1;\ \nu_{k_1} = j_1;\ \nu_{k_2} = j_2\} \cdot P\{s_{j_2} < a_2\} = P\{s_{k_2}' < a_2\} \cdot \sum_{j=1}^{\infty} P\{s_j < a_1;\ \nu_{k_1} = j;\ \nu_{k_2} > j\} =$$

$$= P\{s_{k_2}' < a_2\} \cdot \sum_{j=1}^{\infty} P\{\nu_{k_1} = j;\ \nu_{k_2} > j\} \cdot P\{s_j < a_1\} = P\{s_{k_1}' < a_1\} \cdot P\{s_{k_2}' < a_2\} \cdot P\{\nu_{k_1} < \nu_{k_2}\}.$$

Similarly,

$$\Sigma_2 = P\{s_{k_1}' < a_1\} \cdot P\{s_{k_2}' < a_2\} \cdot P\{\nu_{k_1} > \nu_{k_2}\}.$$

Thus,

$$P\{s'_{k_1}<a_1;\ s'_{k_2}<a_2\}=P\{s'_{k_1}<a_1\}\cdot P\{s'_{k_2}<a_2\}\left(P\{\nu_{k_1}<\nu_{k_2}\}+P\{\nu_{k_1}>\nu_{k_2}\}\right)=P\{s'_{k_1}<a_1\}\cdot P\{s'_{k_2}<a_2\}$$

Lemma 1 is proved.

LEMMA 2. Let $R\in T_1(\{s_i\})$ $(R\in T_2(\{s_i\}))$. Then $R\in T_2(\{s'_i\})$ $(R\in T_1(\{s''_i\}))$ and $y_i = y'_i$ a.s., $z_i = z'_i$ a.s., ($y_i = y''_i$ a.s., $z_i = z''_i$ a.s.), where y_i, y'_i, y''_i (z_i, z'_i, z''_i) are the times when servicing of the i-th call begins (ends) in the systems $\langle\{s_i\},1\rangle$ $(\langle\{s_i\},2\rangle)$, $\langle\{s'_i\},2\rangle$, $\langle\{s''_i\},1\rangle$, respectively.

Proof. The proof follows immediately from the definition of the classes T_1, T_2 and Lemma 1.

Let $n\in N$, $(b_1,\ldots,b_n)$ be an arbitrary n-tuple of numbers. We denote by $\pi_n(b_1,\ldots,b_n)=(a_1,\ldots,a_n)$ a permutation of the set $(b_1,\ldots,b_n)$ such that $a_1\le a_2\le\ldots\le a_n$.

LEMMA 3. Let $n\in N$, $R=\{\nu_i\}_{i=1}^{\infty}\in T_2(\{s_i\})$ and $R^{(n)}=\{\nu_i^{(n)}\}_{i=1}^{\infty}=R^{(n)}(R)$ be such that

a) $(\nu_1^{(n)},\ldots,\nu_n^{(n)})=\pi_n(\nu_1,\ldots,\nu_n)$ a.s.;

b) for $i\in N$, $i\geqslant n+1$ $\nu_i^{(n)}=\nu_i$ a.s.

Then $R^{(n)}\in T_2(\{s_i\})$.

Remark 2. This result is also true when the class $T_2(\{s_i\})$ is replaced by $T_1(\{s_i\})$.

Proof. Without loss of generality, we give the proof for n = 2. We verify that for all $l\in N$ and $j_1,\ldots,j_l$ ($j_k\in N$ for $1\le k\le l$) the event $\{\nu_{j_1}^{(2)}=1,\ldots,\nu_{j_l}^{(2)}=l\}$ does not depend on the σ-algebra $\sigma\{s_r;\ r\ge l\}$ generated by the random variables $\{s_r;\ r\ge l\}$. Indeed, if $1,2\in\{j_1,\ldots,j_l\}$ or $1,2\notin\{j_1,\ldots,j_l\}$, the assertion is obvious. Let $1\in\{j_1,\ldots,j_l\}$, $2\notin\{j_1,\ldots,j_l\}$, $j_q=1$ for some q, $1\le q\le l$ (the case $1\notin\{j_1,\ldots,j_l\}$; $2\in\{j_1,\ldots,j_l\}$ is considered analogously). Then

$$\begin{aligned}&\{\nu_{j_1}^{(2)}=1,\ldots,\nu_{j_{q-1}}^{(2)}=q-1;\ \nu_1^{(2)}=q;\ \nu_{j_{q+1}}^{(2)}=q+1,\ldots,\nu_{j_l}^{(2)}=l\}=\\&=\{\nu_{j_1}=1,\ldots,\nu_{j_{q-1}}=q-1;\ \nu_1=q;\ \nu_{j_{q+1}}=q+1,\ldots,\nu_{j_l}=l\}\cup\\&\cup\{\nu_{j_1}=1,\ldots,\nu_{j_{q-1}}=q-1;\ \nu_2=q;\ \nu_{j_{q+1}}=q+1,\ldots,\nu_{j_l}=l\}.\end{aligned}$$

Since $R\in T_2(\{s_i\})$, the proof of Lemma 3 is complete.

4. Formulation and Proof of Theorems and Corollaries. For $n\in N$, let $F^{(n)}=\{f\,|\,f\colon E_+^n\to E$ and f is Borel; for every n-tuple $(a_1,\ldots,a_n)\in E_+^n$ and any permutation $(b_1,\ldots,b_n)$, assume that $f(a_1,\ldots,a_n)=f(b_1,\ldots,b_n)$; if $(a_1,\ldots,a_n)\le(c_1,\ldots,c_n)$, then $f(a_1,\ldots,a_n)\le f(c_1,\ldots,c_n)\}$.

For $i\in N$, let $y_i^{(0)}$ ($z_i^{(0)}$) be the time when servicing of the i-th call for the FCFS strategy $R^{(0)}$ begins (ends).

THEOREM 2. For every $n\in N$, $f\in F^{(n)}$, $a\geqslant 0$, $R\in T_1(\{s_i\})(R\in T_2(\{s_i\}))$ in the system $\langle\{s_i\},1\rangle$ $(\langle\{s_i\},2\rangle)$, we have the inequalities

$$\begin{aligned}&P\{f(y_1,\ldots,y_n)<a\}\leqslant P\{f(y_1^{(0)},\ldots,y_n^{(0)})<a\};\\&P\{f(z_1,\ldots,z_n)<a\}\leqslant P\{f(z_1^{(0)},\ldots,z_n^{(0)})<a\}.\end{aligned}\tag{6}$$

Remark 3. Since $\max\limits_{1\leqslant k\leqslant m}w_{n,k}=\max\limits_{1\leqslant i\leqslant n}z_i-x_n$ and $f(a_1,\ldots,a_n)=\max\limits_{1\leqslant i\leqslant n}a_i\in F^{(n)}$, Theorem 1 is a consequence of Theorem 2.

Proof of Theorem 2. We introduce the following notation: for $i\in N$, y_i^{α} (z_i^{α}) is the time when servicing of the i-th call starts (ends); r_i^{α} is the number of the channel in which the i-th call is serviced; and $w_{i,l}^{\alpha}$ is the time when the l-th channel is no longer involved in servicing any calls with numbers $\le$ i using servicing strategy R^{α} (here α is some index).

We prove the theorem for $R\in T_2(\{s_i\})$.

Remark 4. For all $n\in N$ and $f\in F^{(n)}$, the distribution of the functionals $f(y_1^{(0)},\ldots,y_n^{(0)})$ and $f(z_1^{(0)},\ldots,z_n^{(0)})$ does not depend on the type of the queueing system.

Remark 5. If the theorem holds for every $R\in T_2(\{s_i\})$, then by Lemma 2 and Remark 4 it is valid for every $R\in T_1(\{s_i\})$.

Fix $n \in N$. We define a service strategy $R^{(n,0)}$ defined by $R = \{\nu_i\}_{i=1}^{\infty} \in T_2(\{s_i\})$ as follows. Calls with numbers i, i > n, are serviced in the same way as in R (i.e., in the same channels, with servicing starting and ending at the same times), and if we write $(d_1, \ldots, d_n) = \pi_n(y_1, \ldots, y_n)$, where $\{y_i\}_{i=1}^{n}$ are the times at which servicing begins under strategy R, calls with numbers 1, 2, . . . , n begin to be serviced at times $d_1, d_2, \ldots, d_n$, respectively, i.e., "with delay." By definition

$$\left(y_1^{(n,0)}, \ldots, y_n^{(n,0)}\right) = (d_1, \ldots, d_n) = \pi_n(y_1, \ldots, y_n) \text{ a.s.} \tag{7}$$

$$\text{and } \pi_n\left(z_1^{(n,0)}, \ldots, z_n^{(n,0)}\right) = \pi_n(z_1, \ldots, z_n) \text{ a.s.} \tag{8}$$

since we consider a system $\langle\{s_i\}, 2\rangle$. Queueing of calls under $R^{(n,0)}$ is the same as for $R^{(n)} = \{\nu_i^{(n)}\}_{i=1}^{\infty}$.

We write $s_i^{(n)} = s_i''(R^{(n)}) = s_{\check{\nu}(n)}$; consider the system $\langle\{s_i^{(n)}\}, 1\rangle$. We remark that by Lemma 2, the random vectors $(y_1^{(n,0)}, \ldots, y_n^{(n,0)})$ and $(y_1'^{(n,0)}, \ldots, y_n'^{(n,0)})$ obtained using strategy $R^{(n,0)}$ in the systems $\langle\{s_i\}, 2\rangle$ and $\langle\{s_i^{(n)}\}, 1\rangle$, respectively, coincide a.s. [the same is true for the vectors $(z_1^{(n,0)}, \ldots, z_n^{(n,0)})$ and $(z_1'^{(n,0)}, \ldots, z_n'^{(n,0)})$; $(r_1^{(n,0)}, \ldots, r_n^{(n,0)})$ and $(r_1'^{(n,0)}, \ldots, r_n'^{(n,0)})$].

Remark 6. Since in Lemma 3 $R^{(n)} \in T_2(\{s_i\})$, for $1 \le i \le n$ the random vectors $(r_i^{(n,0)}, s_1^{(n)}, \ldots, s_{i-1}^{(n)})$ and $(s_i^{(n)}, s_{i+1}^{(n)}, \ldots, s_n^{(n)})$ are independent.

Let $\{s_i^{(n,0)}\}_{i=1}^{\infty}$ be a sequence of random vectors such that

1) all the $s_i^{(n,0)}$ have the same distribution as s_1;

2) the $\{s_i^{(n,0)}\}_{i=1}^{\infty}$ are jointly independent;

3) for $1 \le i \le n$ $s_i^{(n,0)} = s_i^{(n)}$ a.s.;

4) for $1 \le i \le n$ the sets of random variables $\{r_i^{(n,0)}; s_1^{(n,0)}, \ldots, s_{i-1}^{(n,0)}\}$ and $\{s_i^{(n,0)}, s_{i+1}^{(n,0)}, \ldots, s_n^{(n,0)}, \ldots\}$ are mutually independent. By Remark 6, there exists a sequence $\{s_i^{(n,0)}\}$ with these properties.

Consider the queueing system $\langle\{s_i^{(n,0)}\}, 1\rangle$ and service strategy $R^{(n,1)}$, which calls are serviced in order of arrival, but the first n calls begin to be serviced at times $y_1^{(n,0)}, \ldots, y_n^{(n,0)}$ in channels with numbers $r_1^{(n,0)}, \ldots, r_n^{(n,0)}$, respectively, i.e., "with delays." By definition,

$$\left(y_1^{(n,1)}, \ldots, y_n^{(n,1)}\right) = \left(y_1^{(n,0)}, \ldots, y_n^{(n,0)}\right), \tag{9}$$

$$\left(z_1^{(n,1)}, \ldots, z_n^{(n,1)}\right) = \left(z_1^{(n,0)}, \ldots, z_n^{(n,0)}\right) \text{ a.s.} \tag{10}$$

Consider the service strategy $R^{(n,2)}$ in the system $\langle\{s_i^{(n,0)}\}, 1\rangle$ defined as follows. The first n calls for servicing arrive at the same channels as for $R^{(n,1)}$, but without "delays," the remaining calls being serviced after the first n in order of arrival and without "delays." $R^{(n,2)}$ is constructed as follows: for $1 \le l \le m$ put $w_{0,l}^{(n,2)} = w_{0l}$; assume that $w_{k,l}^{(n,2)}$ $(1 \le l \le m)$ have been defined for $1 \le k \le n$; then we put $y_{k+1}^{(n,2)} = \max\left(x_{k+1}; w_{k,r_{k+1}^{(n,0)}}^{(n,2)}\right)$; $w_{k+1,l}^{(n,2)} = w_{k,l}^{(n,2)}$ for $l \ne r_{k+1}^{(n,0)}$; $1 \leqslant l \leqslant m$ and $w_{k+1,r_{k+1}^{(n,0)}}^{(n,2)} = y_{k+1}^{(n,2)} + s_{k+1}^{(n)} = z_{k+1}^{(n,2)}$.

We remark that

$$\left(y_1^{(n,2)}, \ldots, y_n^{(n,2)}\right) \leqslant \left(y_1^{(n,1)}, \ldots, y_n^{(n,1)}\right) \text{ a.s.}, \tag{11}$$

$$\left(z_1^{(n,2)}, \ldots, z_n^{(n,2)}\right) \leqslant \left(z_1^{(n,1)}, \ldots, z_n^{(n,1)}\right) \text{ a.s.}, \tag{12}$$

Note that the strategy $R^{(n,2)}$ is uniquely determined by $\{r_i^{(n,2)}\}_{i=1}^{n}$ and satisfies condition 2), so that Theorem 2 in [1] applies and says that for any $f \in F^{(n)}$, $a \in E_+$

$$P\left\{f\left(y_1^{(n,2)}, \ldots, y_n^{(n,2)}\right) < a\right\} \leqslant P\left\{f\left(y_1^{(0)}, \ldots, y_n^{(0)}\right) < a\right\}; \tag{13}$$

$$P\left\{f\left(z_1^{(n,2)}, \ldots, z_n^{(n,2)}\right) < a\right\} \leqslant P\left\{f\left(z_1^{(0)}, \ldots, z_n^{(0)}\right) < a\right\}. \tag{14}$$

Using Remarks 4-5 and relations (7)-(14), we obtain the assertion of the theorem.

COROLLARY 1. For any $n \in N$, $R \in T_1(\{s_i\})$ $(R \in T_2(\{s_i\}))$, $a \geqslant 0$, the inequalities $P\left\{\frac{1}{n}\sum_{i=1}^{n} b_i^{(0)} < a\right\} \geqslant P\left\{\frac{1}{n} \times \sum_{i=1}^{n} b_i < a\right\}$ hold in the systems $\langle\{s_i\}, 1\rangle$ $(\langle\{s_i\}, 2\rangle)$ for a) $b_i = z_i$; b) $b_i = y_i$; c) $b_i = z_i - x_i$.

COROLLARY 2. Let $\{A_n\}_{n=1}^{\infty}$ be the full group of events on $\langle \Omega, F\rangle$ belonging to the σ-algebra generated by the random variables $\{\tau_i\}_{i=1}^{\infty}$, and let

$$\xi(\omega, a_1, \ldots, a_n, \ldots) = \sum_{n=1}^{\infty} I\{A_n\} \cdot f_n(a_1, \ldots, a_n) \quad \text{for} \quad (a_1, \ldots, a_n, \ldots) \in E_+^{\infty}, \quad f_n \in F^{(n)}, \qquad \omega \in \Omega.$$

Then for $R \in T_1(\{s_i\})$ $(R \in T_2(\{s_i\}))$ we have

$$P\{\xi(\omega, z_1^{(0)}, \ldots, z_n^{(0)}, \ldots) < a\} \geqslant P\{\xi(\omega, z_1, \ldots, z_n, \ldots) < a\}$$

in the system $\langle\{s_i\}, 1\rangle$ $(\{s_i\}, 2\rangle)$ for every $a \geq 0$.

For $t \geq 0$ let $\gamma(t, R)$ be the time needed for the system to process calls arriving prior to time t using R. For $n \geq 1$ we put $A_n = \{x_n \leq t; x_{n+1} > t\}$. Then on the set A_n,

$$\gamma(t, R) = \max_{1 \leqslant i \leqslant n} z_i - t$$

and by Corollary 2, we obtain

COROLLARY 3. For k = 1, 2 and any $R \in T_k(\{s_i\})$, $t \geqslant 0$; $a \geqslant 0$, the inequality

$$P\{\gamma(t, R^{(0)}) < a\} \geqslant P\{\gamma(t, R) < a\}$$

holds in the system $\langle\{s_i\}, k\rangle$.

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SELF-ADJOINTNESS OF THE SCHRÖDINGER OPERATOR WITH AN INFINITE NUMBER OF VARIABLES

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Self-adjoint differential operators with an infinite number of independent variables have been studied in [1-5]. In particular, in [2] one has proved the essential self-adjointness of the infinite-dimensional Schrödinger operator, whose potential is twice differentiable and semibounded on some "admissible" set. A similar fact has been proved in [5] under the condition that the potential admits a sufficiently fast approximation by cylindrical functions which have a special estimate from below. In the present paper one establishes the essential self-adjointness of an infinite-dimensional semibounded Schrödinger operator whose potential satisfies only measurability and boundedness conditions on any bounded sets.

1. Let H be a real, separable Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$, equipped [1] with the Hilbert spaces B and $B^*: B^* \subset H \subset B$ so that the embedding operator from H into B is a Hilbert–Schmidt operator. In the space B we define the Gaussian measure $\mu(dx)$ by the characteristic functional $\chi(\varphi) = \exp(-{}^1/_2\|\varphi\|^2)$, $(\varphi \in H)$ [1]. The set of all real μ-measurable functions $u(x)$ on B, for which $|u(x)|^2$ is integrable with respect to the measure $\mu(dx)$, will be denoted by $L_2(B)$. The value of the linear functional $e \in B^*$ at the element $x \in B$ will be denoted by (e, x).

We denote by $C^{\infty}(B)$ the set of all infinitely differentiable (Frechet) real functions on B, bounded on any bounded set from B together with the derivatives of any order. Let $C_0^{\infty}(B)$ be the set of all the functions from $C^{\infty}(B)$ which have in B a bounded support. We define convergence in $C_0^{\infty}(B)$ in the following manner: $\varphi_n \in C_0^{\infty}(B)$

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