THE LINEARIZED PROBLEM OF MAGNETO-PHOTOELASTICITY

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ABSTRACT. The equations of magneto-photoelasticity are derived for a nonhomogeneous background isotropic medium and for a variable gyration vector. They coincide with Aben’s equations in the case of a homogeneous background medium and of a constant gyration vector. We obtain an explicit linearized formula for the fundamental solution under the assumption that variable coefficients of equations are sufficiently small. Then we consider the inverse problem of recovering the variable coefficients from the results of polarization measurements known for several values of the gyration vector. We demonstrate that the data can be easily transformed to a family of Fourier coefficients of the unknown function if the modulus of the gyration vector is agreed with the ray length.

1. Introduction

Let a plane polarized electromagnetic wave of frequency $\omega$ propagate in a homogeneous isotropic dielectric medium. If a constant exterior magnetic field (gyration vector) is applied to the medium, which is parallel to the wave propagation direction, then the polarization plane rotates along a ray. The rotation velocity is proportional to the modulus of the gyration vector and to the frequency $\omega$. This phenomenon was discovered by Faraday in 1845 and was named the Faraday rotation. An elegant explanation of the phenomenon was given by Maxwell, see for example [6, §101].

In 1970, Aben [1] proposed to use the Faraday rotation in photoelasticity and introduced the term magneto-photoelasticity. He wrote down differential equations for evolution of the light polarization along a ray and analyzed the equations in the partial case of constant quasi-principle directions. The approach was discussed in a number of subsequent papers (see [3, 8] and references there) but remained to be a hypothetic method till the last decade. Recently, a measurement instrument was designed at the University of Sheffield which combined an optical polariscope and solenoidal coils [4].

In our opinion, the mathematical nature of magneto-photoelasticity is still not well understood. Indeed, Aben has analyzed the very partial case of constant quasi-principle directions when the classical Wertheim law holds. The general case is much more complicated. We hope the present article fills in the gap in the mathematical foundation of magneto-photoelasticity.

In our opinion, the mathematical basis of general photoelasticity, as presented in Part 1 of Aben’s book [2], is also not quite irreproachable mathematically. The resulting equations are okey, but some arguments used in their derivation seem to be not quite correct. An alternative approach for deriving equations of photoelasticity
was proposed by the author in [9] and then presented in details in [10, Section 5.1]. The approach is based on the quasi-isotropic approximation of geometric optics. The method was first proposed by Kravtsov [5].

In Section 2, we use the same quasi-isotropic approximation for deriving the equations of magneto-photoelasticity in the case of a nonhomogeneous background medium and of a variable gyration vector. To author’s knowledge, the equations were not known before in such a generality. We actually demonstrate that these equations coincide with the Rytov law for quasi-isotropic gyrotrropic media. In the case of a pure gyrotrropic medium, i.e., when there is no other medium anisotropy, the equations have an obvious solution that represents the Faraday rotation. At the end of the section, we consider the case of a homogeneous background medium and of a constant gyration vector to obtain Aben’s equations of magneto-photoelasticity.

In the general case, the solution to Aben’s equations depends on equation coefficients in a very complicated way. To simplify the analysis of the solution, we assume the variable coefficients to be sufficiently small such that higher order terms can be ignored. Such a linearization works well in the most cases of traditional photoelasticity, see [9] for example. So, we hope it will work in magneto-photoelasticity as well. In Section 3, we derive an explicit linearized formula for the solution to Aben’s equations.

In Section 4, we consider the inverse problem of recovering the medium anisotropy (stresses in particular) from the results of polarization measurements that are known for several values of the gyration vector. First of all, only two components of the dielectric permeability tensor participate in Aben’s equations. Therefore we consider the problem of recovering that components. If the modulus of the gyration vector is agreed with the sample thickness, then the polarization measurement gives us exactly a Fourier coefficient of the sought function. A number of such measurements gives us several Fourier coefficients and we can recover the function approximately as a partial sum of the Fourier series. All questions on accuracy of the recovering are trivial in this approach.

2. ELECTROMAGNETIC WAVES IN QUASI-ISOTROPIC GYROTROPIC MEDIA

We consider the Maxwell equations for time-harmonic waves of frequency $\omega$

\begin{align*}
\text{curl}\ H + ikD &= 0, \\
\text{curl}\ E - ikH &= 0,
\end{align*}

where $k = \omega/c$ is the wave number, with the material equation (in Cartesian coordinates)

\begin{equation}
D_j = \varepsilon_{jk}E_k.
\end{equation}

We assume the magnetic permeability to be identically unit and the dielectric permeability tensor to be represented in the form

\begin{equation}
\varepsilon_{jk} = n^2\delta_{jk} + \frac{1}{k}\chi_{jk},
\end{equation}
where \( (\delta_{jk}) \) is the Kronecker tensor and \( n = n(x) \) is a positive sufficiently smooth function of a point \( x \in \mathbb{R}^3 \) (the refraction coefficient). The small factor \( 1/k \) is written at the second term on the right-hand side of (2.3) to emphasize that the term is considered as a small anisotropic perturbation of the background isotropic medium with the refraction coefficient \( n \). Equation (2.3) was first introduced by Kravtsov [5] who used the name quasi-isotropic media for such media.

Next, we assume the tensor \( \chi \) to have the form

\[
\chi_{jk} = S_{jk} - i e_{jkl} g_l,
\]

where \( (S_{jk}) = (S_{jk}(x)) \) is a real symmetric tensor (i.e. \( S_{jk} = S_{kj} \)) smoothly depending on a point \( x \); \( g = g(x) \) is a real vector smoothly depending on a point which is called the gyration vector; \( (e_{jkl}) \) is the skew-symmetric tensor satisfying \( e_{123} = 1 \) in positively oriented Cartesian coordinates (the discriminant tensor); and \( i \) is the imaginary unit. Observe that \( \chi \) is a Hermitian tensor, i.e., \( \chi_{jk} = \bar{\chi}_{kj} \). In particular, all results obtained in [10, Section 5.1] are valid in our case. Substitution of (2.3) and (2.4) into (2.2) gives

\[
D = n^2 E + \frac{1}{k} SE - \frac{i}{k} g \times E,
\]

where \( \times \) stands for the vector product.

In photoelasticity, the tensor field \( S(x) \) is caused by mechanical stresses (the photoelastic phenomenon). In the most cases this tensor is very small because of the smallness of the photoelastic material parameter. But in principle, \( S \) can be caused by any other small medium anisotropy.

The method of geometric optics (or the ray method) consists of representation of each of vector fields \( A = E, H, D \) by the asymptotic series of the type

\[
A(x) = e^{ik\tau(x)} \sum_{m=0}^{\infty} \frac{A^{(m)}(x)}{(ik)^m},
\]

In the most cases the zeroth terms \( E^{(0)}, H^{(0)} \) give a good approximation to the exact solution. We insert the series into equations (2.1)–(2.2), implement differentiations and equate the coefficients at the same powers of the wave number \( k \) on the left- and right-hand sides of the so obtained equalities. In such a way we arrive at an infinite system of equations that can be, in principle, solved recursively to obtain all terms of the asymptotic series. Here, we present very briefly the main results of such an analysis that is presented in details in Section 5.1 of [10].

First of all, the initial equations (zeroth equations) of the infinite system do not involve the tensor \( \chi \) just due to the factor \( 1/k \) in (2.3). The zeroth equations determine geometry of rays that is thus the same for the quasi-isotropic medium and for the background isotropic medium. The phase \( \tau(x) \) satisfies the eikonal equation

\[
|\nabla \tau|^2 = n^2
\]

and light rays are geodesics of the Riemannian metric

\[
d\tau^2 = n^2 |dx|^2.
\]
We use the notations $|a|_e$ and $\langle a, b \rangle_e$ for the Euclidean norm and scalar product, while $|a|$ and $\langle a, b \rangle$ denote the norm and scalar product with respect to the Riemannian metric (2.5). The vectors $E^{(0)}$ and $H^{(0)}$ are orthogonal to a ray, and the amplitude $A = |E^{(0)}|_e$ varies along the ray as

$$A = \frac{C}{\sqrt{nJ}},$$

where $C$ is a constant for a given ray and $J$ is the geometric spreading, i.e., the area of the cross-section of the ray tube. The formula has the clear physical sense: the energy propagates along ray tubes in the scope of the zeroth approximation.

The polarization vector $\eta$ is defined by $\eta = \frac{1}{nA}E^{(0)}$. The factor $1/n$ is involved to make $\eta$ a unit vector in metric (2.5): $|\eta| = 1$. Along a ray $\gamma = \gamma(\tau)$ parameterized by the eikonal, the polarization vector satisfies the Rytov law for quasi-isotropic media:

$$\frac{D \eta}{d\tau} = \frac{i}{2n^2} \hat{P}_\gamma \chi \eta,$$

where $\hat{\gamma}$ is the speed vector of the ray and $\hat{P}_\gamma$ is the orthogonal projection onto the polarization plane (that is the plane orthogonal to $\hat{\gamma}$). The Riemannian metric (2.5) is involved into (2.6) through the differential operator $D/d\tau = \hat{\gamma}^k \nabla_k$, where $\nabla$ is the covariant derivative with respect to metric (2.5).

The main equation of magneto-photoelasticity is obtained by substituting value (2.4) for the tensor $\chi$ into the Rytov law (2.6):

$$\frac{D \eta}{d\tau} = \frac{i}{2n^2} \hat{P}_\gamma (S\eta - i g \times \eta),$$

This can be written also in the form

$$\frac{D \eta}{d\tau} = \frac{i}{2n^2} \hat{P}_\gamma S\eta + \frac{\langle g, \hat{\gamma} \rangle}{2n^3} \eta^\perp,$$

where $\eta^\perp$ is the result of rotating the vector $\eta$ by the right angle in the positive direction. The rotation is done in the polarization plane that is oriented so that, for a positive plane frame $(e_1, e_2)$, the frame $(e_1, e_2, \hat{\gamma})$ is positive in $\mathbb{R}^3$. As compared with (2.7), equation (2.8) has two advantages: (1) it emphasizes that the gyration vector $g$ is involved through the scalar product $\langle g, \hat{\gamma} \rangle$ only and (2) equation (2.8) is written completely in terms of the Riemannian metric (2.5) while the vector product in (2.7) is still understood in the Euclidean sense.

Being written in coordinates, (2.8) is a linear system of ordinary differential equations along a ray with a skew-Hermitian matrix. Therefore the norm $|\eta(\tau)|$ is constant for a solution.

Let us consider the case of a pure gyrotryic medium, i.e., the case when $S$ is identically zero

$$\frac{D \eta}{d\tau} = \frac{\langle g, \hat{\gamma} \rangle}{2n^3} \eta^\perp.$$
This equation can be easily solved. Indeed, choose an orthonormal basis \((e_1(\tau), e_2(\tau))\) of the polarization plane along a ray which is parallel along the ray, i.e., \(De_j/d\tau = 0\) \((j = 1, 2)\). Then the general solution to (2.9) can be written as

\[
\eta(\tau) = C\left( \cos \varphi(\tau)e_1(\tau) + \sin \varphi(\tau)e_2(\tau) \right),
\]

where \(C\) is a constant and

\[
\varphi(\tau) = \varphi_0 + \int_0^\tau \frac{\langle g, \dot{\gamma} \rangle}{2n^3} d\tau,
\]

the integration is done over the ray \(\gamma\). Geometrically, (2.10) is the rotation with the velocity \(\langle g, \dot{\gamma} \rangle/2n^3\). Thus, the second term on the right-hand side of (2.8) is responsible for the Faraday rotation.

The polarization vector \(\eta\) is a complex 2-vector satisfying \(|\eta| = 1\). Hence \(\eta\) is described by three real parameters. As known, such parameters can be chosen as \(\varphi, e, \psi\) where \(\varphi\) is the wave phase, \(e\) is the eccentricity of the polarization ellipse, and \(\psi\) is the angle between the major axis of the ellipse and a fixed direction on the polarization plane. Only the polarization parameters \(e, \psi\) are measured in practical polarimetry, i.e., the shape of the polarization ellipse and its disposition. (In practice, the measured parameters are the optical retardation \(\Delta\) and isocline parameter \(\kappa\); the pair \((\Delta, \kappa)\) is in one-to-one correspondence with \((e, \psi)\).) As for the phase \(\varphi\) is concerned, measuring this quantity relates to measuring distances that are comparable with the wavelength; therefore the phase is not usually measured.

The dependence on the phase \(\varphi\) is eliminated by transforming equation (2.8) as follows. For a given ray \(\gamma(\tau)\), we choose an orthonormal basis \((e_1(\tau), e_2(\tau))\) of the polarization plane along the ray as above, i.e., \(De_j/d\tau = 0\) \((j = 1, 2)\), and such that \((e_1(\tau), e_2(\tau), \dot{\gamma}(\tau))\) is a positive frame of \(\mathbb{R}^3\). A solution to equation (2.8) can be written as

\[
\eta(\tau) = \eta_1(\tau)e_1(\tau) + \eta_2(\tau)e_2(\tau)
\]

and equation (2.8) is equivalent to the system

\[
\begin{align*}
\frac{d\eta_1}{d\tau} &= \frac{i}{2n^2} (S_{11}\eta_1 + S_{12}\eta_2) - \frac{\langle g, \dot{\gamma} \rangle}{2n^3} \eta_2, \\
\frac{d\eta_2}{d\tau} &= \frac{i}{2n^2} (S_{21}\eta_1 + S_{22}\eta_2) + \frac{\langle g, \dot{\gamma} \rangle}{2n^3} \eta_1.
\end{align*}
\]

A solution to the system satisfies \(|\eta_1(\tau)|^2 + |\eta_2(\tau)|^2 = \text{const}\). As can be easily shown, the complex ratio \(\eta_2/\eta_1\) is in one-to-one correspondence with the polarization parameters \((e, \psi)\), see [10, Section 6.1] for details. In other words, the polarization parameters do not change if a solution \((\eta_1(\tau), \eta_2(\tau))\) is multiplied by \(e^{i\lambda(\tau)}\), where \(\lambda(\tau)\) is an arbitrary real function on the ray. Using this freedom, we change the variables in system (2.11) as

\[
\zeta = \exp \left[ -\frac{i}{4n^2} \int_\tau^{\tau_0} (S_{11} + S_{22}) d\tau \right] \eta.
\]
Then system (2.11) is transformed to the next one

\[
\begin{align*}
\frac{d\zeta_1}{d\tau} &= \frac{i}{2n^2} \left( \frac{1}{2} (S_{11} - S_{22}) \zeta_1 + S_{12} \eta_2 \right) - \frac{\langle g, \dot{\gamma} \rangle}{2n^3} \zeta_2, \\
\frac{d\zeta_2}{d\tau} &= \frac{i}{2n^2} \left( S_{21} \zeta_1 + \frac{1}{2} (S_{22} - S_{11}) \zeta_2 \right) + \frac{\langle g, \dot{\gamma} \rangle}{2n^3} \zeta_1.
\end{align*}
\]

System (2.12) is written in a basis \((e_1(\tau), e_2(\tau), \dot{\gamma}(\tau))\) related to the ray. The invariant form of the system is as follows (we return to denoting a solution by \(\eta\)):

\[
\frac{D\eta}{d\tau} = \frac{i}{2n^2} (Q_\gamma S) \eta + \frac{\langle g, \dot{\gamma} \rangle}{2n^3} \eta^\perp.
\]

Here, for a vector \(0 \neq \xi \in \mathbb{R}^3\), \(Q_\xi\) is the orthogonal projection of the space \(M_3\) of symmetric \(3 \times 3\)-matrices onto the two-dimensional subspace \(\{ A \in M_3 \mid A \xi = 0, \text{tr} A = 0 \}\), where \(M_3\) is endowed with the standard scalar product \(\langle A, B \rangle = \text{tr} (AB^*)\). Being written in coordinates, (2.13) is a linear system of ordinary differential equations with a skew-Hermitian trace-free matrix. See [10, Section 6.1] for details of transforming (2.12) into the invariant form (2.13).

Finally, we consider the case of a homogeneous background medium, i.e., \(n = \text{const}\), and of a constant gyration vector. Let \(xyz\) be Cartesian coordinates in \(\mathbb{R}^3\). We consider electromagnetic waves propagating in the \(z\)-direction. Therefore we choose the eikonal \(\tau(x, y, z) = nz\). The gyration vector should be parallel to the wave propagation direction; we thus choose \(g = 2an(0, 0, 1)\) with some constant \(a\). For the vector \(\dot{\gamma} = n^{-1}(0, 0, 1)\), we have:

\[
Q_\gamma S = \frac{1}{2} \begin{pmatrix}
S_{xx} - S_{yy} & 2S_{xy} & 0 \\
2S_{yx} & S_{yy} - S_{xx} & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Substituting these values into (2.13), we obtain for the polarization vector \(\eta = (\eta_x, \eta_y, 0)\)

\[
\begin{align*}
\frac{d\eta_x}{dz} &= \frac{i}{2n} \left( \frac{1}{2} (S_{xx} - S_{yy}) \eta_x + S_{xy} \eta_y \right) - a \eta_y, \\
\frac{d\eta_y}{dz} &= \frac{i}{2n} \left( S_{yx} \eta_x + \frac{1}{2} (S_{yy} - S_{xx}) \eta_y \right) + a \eta_x.
\end{align*}
\]

This coincides, up to non-relevant details, with Aben’s equations [1].

We are going to write (2.14) in a matrix form. To this end we define two functions

\[
\alpha(z) = \frac{1}{4n} (S_{xx}(z) - S_{yy}(z)), \quad \beta(z) = \frac{1}{2n} S_{xy}(z).
\]

The functions depend actually on all variables \((x, y, z)\) but the dependence on \((x, y)\) is not designated explicitly since these variables will be fixed (all arguments will deal with a fixed ray). We introduce also three \(2 \times 2\)-matrices

\[
\begin{align*}
J &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}
\end{align*}
\]
and re-denote the polarization vector as \( \eta = (\eta_x, \eta_y) \). Then (2.14) can be written as

(2.17) \[ \frac{d\eta}{dz} = A\eta \]

with the matrix

(2.18) \[ A(z) = i\alpha(z)K + i\beta(z)L - aJ. \]

Let \( U(z) \) be the fundamental matrix of equation (2.17) so that

\[ \eta(z) = U(z)\eta(0). \]

This matrix solves the initial value problem

(2.19) \[ \frac{dU}{dz} = AU, \quad U(0) = I, \]

where \( I \) is the unit matrix. In what follows, we study the matrix problem (2.19), this is equivalent to studying the initial value problem for equation (2.17) with an arbitrary \( \eta(0) \). The matrix \( U(z) \) belongs to the group \( SU(2) \) of unitary matrices with unit determinant since \( A(z) \) is a skew-Hermitian matrix with zero trace. This observation actually goes back to Poincaré [7]: polarization phenomena are governed by the group \( SU(2) \).

In the case of \( \alpha = \beta \equiv 0 \), the solution to the initial value problem (2.19)

\[ U(z) = \begin{pmatrix} \cos(az) & -\sin(az) \\ \sin(az) & \cos(az) \end{pmatrix} \]

is the rotation by the angle \( \theta = az \). Comparing this with the known relation \( \theta = \nu Hz \), where \( \nu \) is the Verdet constant, we obtain the relation between \( a \) and the intensity \( H \) of the exterior magnetic field

(2.20) \[ a = \nu H. \]

3. The linearized solution to the initial value problem

By (2.18), the matrix \( A(z) \) depends on two functions \( \alpha(z) \) and \( \beta(z) \). In the next section, we will study the inverse problem of recovering the functions \( \alpha(z) \) and \( \beta(z) \) from the output value \( U(Z) \) of the solution to problem (2.19). To this end we have first of all to investigate the dependence of the solution \( U(z) \) on the functions \( (\alpha(z), \beta(z)) \). In the general case the dependence is very complicated. We simplify the problem by assuming \( \alpha \) and \( \beta \) to be sufficiently small so that we can ignore terms that depend on \( (\alpha, \beta) \) quadratically, cubically, and so on. For brevity, such terms are called higher order terms in what follows.

Introducing the matrix

(3.1) \[ H(z) = i(\alpha(z)K + \beta(z)L), \]

we rewrite (2.18) in the form

(3.2) \[ A(z) = H(z) - aJ, \]
We are looking for the solution to the initial value problem (2.19) in the form
\[(3.3) \quad U(z) = e^{-az}V(z).\]

Then \(V\) solves the problem
\[(3.4) \quad \frac{dV}{dz} = BV, \quad V(0) = I\]
with the matrix
\[(3.5) \quad B(z) = e^{az}A(z)e^{-az} + aJ = e^{az}H(z)e^{-az}.\]

The matrices \(K\) and \(L\) anti-commute with \(J\), more precisely
\[(3.6) \quad KJ = -JK = L, \quad JL = -LJ = K.\]

This implies, for arbitrary \(z\) and \(t\),
\[(3.7) \quad H(t)e^{az} = e^{-az}H(t).\]

Therefore (3.5) is simplified to the following:
\[(3.8) \quad B(z) = e^{2az}H(z).\]

We rewrite (3.4) as the integral equation
\[V(z) = I + \int_0^z B(t)V(t)\, dt.\]

The iterates of the Volterra integral equation always converge. We have
\[(3.9) \quad V(z) = I + \sum_{k=0}^\infty V_k(z),\]
where
\[(3.10) \quad V_k(z) = \int_0^z dt_0 \int_0^{t_0} dt_1 \ldots \int_0^{t_{k-1}} dt_k B(t_0)B(t_1)\ldots B(t_k).\]

By (3.1) and (3.8), the matrix \(B\) depends linearly on \((\alpha, \beta)\). Therefore \(V_0\) is linear in \((\alpha, \beta)\) while other summands of the series in (3.9) should be considered as higher order terms. So, we rewrite (3.9) in the form
\[(3.11) \quad V(z) = I + \int_0^z e^{2at}H(t)\, dt + R(z),\]
where
\[(3.12) \quad R(z) = \sum_{k=1}^\infty V_k(z).\]

The formula
\[(3.13) \quad e^{tJ} = I \cos t + J \sin t\]
holds for any $t \in \mathbb{R}$. It is easily proved on the base of the relation $J^2 = -I$ by analogy with Euler’s formula $e^{it} = \cos t + i \sin t$. On using (3.1), (3.6), and (3.13), we obtain
\[ e^{2atJ} H(t) = iK \left( \cos(2at)\alpha(t) + \sin(2at)\beta(t) \right) + iL \left( -\sin(2at)\alpha(t) + \cos(2at)\beta(t) \right). \]
Substitute this into (3.11) to obtain the final formula
\[ e^{azJ} U(z) = V(z) = I + iK \int_0^z \left( \cos(2at)\alpha(t) + \sin(2at)\beta(t) \right) dt \]
(3.14)
\[ + iL \int_0^z \left( -\sin(2at)\alpha(t) + \cos(2at)\beta(t) \right) dt + R(z). \]
Since both the integrals on the right-hand side of (3.14) are real, $U(z) - e^{-azJ}$ must be a pure imaginary matrix up to higher order terms.

Now, we are going to estimate the remainder term $R(z)$ on the right-hand side of (3.14). We use the norm
\[ \|C\| = \max\{|c_{11}| + |c_{12}|, |c_{21}| + |c_{22}|\} \quad \text{for} \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \]
on the space of $2 \times 2$-matrices. It is agreed with the matrix product: $\|CD\| \leq \|C\| \|D\|$. As is seen from (2.16) and (3.1),
\[ (3.15) \quad h(z) := \|H(z)\| = |\alpha(z)| + |\beta(z)|. \]
From (3.13),
\[ (3.16) \quad \|e^{tf}\| = |\cos t| + |\sin t| \leq \sqrt{2} \quad (t \in \mathbb{R}). \]
By (3.7),
\[ B(t_0)B(t_1) \ldots B(t_k) = e^{2at_0J} H(t_0) e^{2at_1J} H(t_1) \ldots e^{2at_kJ} H(t_k). \]
With the help of the commutator formula (3.8), this is transformed to the form
\[ B(t_0)B(t_1) \ldots B(t_k) = \exp \left( 2a(t_0 - t_1 + \cdots + (-1)^k t_k)J \right) H(t_0)H(t_1) \ldots H(t_k). \]
On using (3.15) and (3.16), we derive the inequality
\[ \|B(t_0)B(t_1) \ldots B(t_k)\| \leq \sqrt{2} h(t_0) h(t_1) \ldots h(t_k) \]
and estimate integral (3.10) as follows:
\[ (3.17) \quad \|V_k(z)\| \leq \sqrt{2} \int_0^z dt_0 \int_0^{t_0} dt_1 \ldots \int_0^{t_{k-1}} dt_k h(t_0) h(t_1) \ldots h(t_k) = \sqrt{2} \frac{1}{(k+1)!} \left( \int_0^z h(t) dt \right)^{k+1}. \]
The last equality on (3.17) holds for any function $h$ and can be easily proved by induction in $k$. Indeed, writing down the equality for $k = \ell$ and differentiating this equality with respect to $z$, we obtain the same equality for $k = \ell - 1$. This justifies the induction step.
Now we are able to estimate the remainder \( R(z) \) that is defined by (3.12). Let us introduce the notation

\[
\delta = \delta(z) := \int_0^z h(t) \, dt = \|\alpha\|_{L^1(0,z)} + \|\beta\|_{L^1(0,z)}.
\]

By (3.12) and (3.17),

\[
\|R(z)\| \leq \sum_{k=1}^{\infty} \|V_k(z)\| \leq \sqrt{2} \sum_{k=2}^{\infty} \frac{\delta_k}{(k+2)!} \leq \frac{1}{\sqrt{2}} \delta^2 \sum_{k=0}^{\infty} \frac{\delta_k}{k!} = \sqrt{2} \delta^2.
\]

Let us formulate the main result of the current section as follows:

**Proposition 3.1.** Let \( \alpha, \beta \in L^1(0,Z) \) \((0 < Z < \infty)\) satisfy

\[
\|\alpha\|_{L^1(0,Z)} + \|\beta\|_{L^1(0,Z)} \leq 1
\]

and \( a \in \mathbb{R} \) be a constant. The solution to the initial value problem (2.19) with the matrix \( A \) given by (2.18) can be represented in the form

\[
U(z) = e^{-azJ} \left[ I + iK \int_0^z \left( \cos(2at)\alpha(t) + \sin(2at)\beta(t) \right) dt \right. \\
+ \left. iL \int_0^z \left( -\sin(2at)\alpha(t) + \cos(2at)\beta(t) \right) dt \right] + R(z)
\]

for every \( z \in [0,Z] \), where the remainder \( R(z) \) admits the estimate

\[
\|R(z)\| \leq e \left( \|\alpha\|_{L^1(0,Z)} + \|\beta\|_{L^1(0,Z)} \right)^2.
\]

4. The inverse problem

Let the medium under investigation be contained in the layer \( 0 \leq z \leq Z \). Measuring the polarization parameters of the transmitted light, we obtain the value \( U(Z) \) of the solution to the initial value problem (2.19). Our problem is to recover the functions \( \alpha(z) \) and \( \beta(z) \) participating in (2.18). Of course the result of one measurement does not give us data for such recovering. We repeat the measurement for several values of the parameter \( a \) participating in (2.18), i.e., for several values of the intensity of the exterior magnetic field. From now on we will use the notation \( U(Z,a) \) instead of \( U(Z) \) in order to designate explicitly the dependence on \( a \). We will show that such a collection of data \( U(z,a_m) \) known for an appropriate set of \( a_m \) allows us to reconstruct \( \alpha(z) \) and \( \beta(z) \) with any prescribed precision.

By Proposition 3.1, our data are represented, up to higher order terms, by the matrix

\[
D(a) = Kd_K(a) + Ld_L(a) = \frac{1}{i} \left( e^{azJ}U(Z,a) - I \right),
\]
where

\begin{align*}
    d_K(a) &= \int_0^Z \left( \cos(2az)\alpha(z) + \sin(2az)\beta(z) \right) dz, \\
    d_L(a) &= \int_0^Z \left( -\sin(2az)\alpha(z) + \cos(2az)\beta(z) \right) dz.
\end{align*}

Since $K$ and $L$ are linearly independent, $d_K(a)$ and $d_L(a)$ are known if $U(Z,a)$ is known.

We remember that $\alpha$ and $\beta$ are real functions. Introducing the complex function

$$
\gamma(z) = \alpha(z) + i\beta(z),
$$

we immediately derive from (4.1) and (4.2)

\begin{equation}
    d(a) := d_K(a) + id_L(a) = \int_0^Z e^{-2iaz} \gamma(z) dz.
\end{equation}

Thus, we are given the value $\hat{\gamma}(2a)$ of the Fourier transform of the function $\gamma$. Such data, known for an appropriate finite sequence of values of the parameter $a$, are sufficient to recover $\gamma$ with any precision. The simplest way of such recovering is presented in the next paragraph.

We transform (4.3) to the integral over the symmetric interval $[-Z, Z]$ by the linear change of the integration variable $z = (z' + Z)/2$

\begin{equation}
    e^{iaZ} d(a) = \frac{1}{2} \int_{-Z}^Z e^{-i\pi m z/Z} f(z) dz.
\end{equation}

where

\begin{equation}
    f(z) = \gamma((z + Z)/2) \quad \text{for} \quad -Z \leq z \leq Z.
\end{equation}

If we choose $a = \pi/Z$, then

$$
\frac{(-1)^m}{Z} d(ma) = \frac{1}{2Z} \int_{-Z}^Z e^{-i\pi mz/Z} f(z) dz
$$

is just the Fourier coefficient of the function $f$ and the function is recovered as the Fourier series

\begin{equation}
    f(z) = \frac{1}{Z} \sum_{m=-\infty}^{\infty} (-1)^m d(ma) e^{i\pi mz/Z}.
\end{equation}

In Fourier series theory, several conditions are known which guarantee the convergence of the Fourier series and the validity of (4.6). The simplest of such conditions which is enough for the most of applications is as follows: $f(z)$ should be a continuous and
piece-wise $C^1$-function on $[0, Z]$, i.e., there exists a partition $0 = z_0 < z_1 < \ldots < z_n = Z$ of the segment $[0, Z]$ such that the restriction $f|_{[z_{k-1}, z_k]}$ is a $C^1$-function for every $1 \leq k \leq n$.

We have thus proved

**Theorem 4.1.** Assume $\alpha(z)$ and $\beta(z)$ to be real continuous and piece-wise $C^1$-functions on a segment $[0, Z]$ $(0 < Z < \infty)$ satisfying (3.19). For $a \in \mathbb{R}$, let $U(Z, a)$ be the value at $z = Z$ of the solution to the initial value problem (2.19) with the matrix $A$ defined by (2.16) and (2.18). Let $d(a) = d_K(a) + id_L(a)$, where

$$Kd_K(a) + Ld_L(a) = \frac{1}{i} \left( e^{aZJ}U(Z, a) - I \right).$$

Choose $a = \pi/Z$ and assume the data $d(ma)$ to be known for every integer $m$. Then the functions $\alpha(z)$ and $\beta(z)$ can be approximately recovered by the formula

$$\alpha(z) + i\beta(z) = \frac{1}{Z} \sum_{m=-\infty}^{\infty} d(ma) e^{2i\pi mz/Z} + R(z) \quad (0 \leq z \leq Z),$$

where the remainder term admits the estimate

$$|R(z)| \leq C \left( \|\alpha\|^2_{L^1(0,Z)} + \|\beta\|^2_{L^1(0,Z)} \right)$$

with some universal constant $C$.

In practice, of course, a finite number of data $d(ma)$ is measured and we use a finite segment of the Fourier series

$$\alpha(z) + i\beta(z) \approx \frac{1}{Z} \sum_{m=-M}^{M} d(ma) e^{2i\pi mz/Z}. \quad (4.8)$$

How large value of $M$ should be chosen to get a desired accuracy? This question is completely investigated in theory of Fourier series.

**REFERENCES**


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