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THE FARKAS LEMMA REVISITED

НОВОСИБИРСК

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The Farkas Lemma is extended to simultaneous linear operator and polyhedral sublinear operator inequalities by Boolean valued analysis.

Keywords and phrases:
Dedekind complete vector lattice, linear programming, linear inequalities, polyhedral sublinear inequalities, Boolean valued model

Кутателадзе С. С.

Новая форма леммы Фаркаша.

С помощью техники булевозначного анализа лемма Фаркаша распространена на системы линейных операторных неравенств и полиэдральных сублинейных операторных неравенств.

Ключевые слова и фразы:
пространства Канторовича, линейное программирование, линейные неравенства, полиэдральные сублинейные неравенства, булевозначные модели

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THE FARKAS LEMMA REVISITED

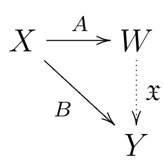
Introduction

The Farkas Lemma, also known as the Farkas–Minkowski Lemma, plays a key role in linear programming and the relevant areas of optimization (cp. [1]). Some rather simple proof of the lemma is given in [2]. Using the technique of Boolean valued analysis (cp. [3]), we abstract the Farkas Lemma to simultaneous linear and polyhedral sublinear inequalities with operators.

Assume that X is a real vector space, Y is a *Kantorovich space* also known as a Dedekind complete vector lattice or a complete Riesz space. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebra of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \leq 0\} := \{x \in X \mid Tx \leq 0\}$ and $\{T = 0\} := \ker(T) := T^{-1}(0)$ for $T : X \to Y$.

1. Simultaneous Linear Inequalities

The Farkas Lemma deals with the following diagram:



We list the general facts about the commutativity of the diagram:
(1) $(\exists \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$;
(2)[1] *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$ then*

$$(\exists \mathfrak{X} \geq 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}.$$

Lemma 1.1. *Let X be a vector space over some subfield R of the reals $\mathbb{R}$. Assume that f and g are R-linear functionals on X; in symbols, $f, g \in X^{\#} := L(X, \mathbb{R})$.*

For the inclusion

$$\{g \leq 0\} \supset \{f \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha \in \mathbb{R}_+$ satisfying $g = \alpha f$.

The author is grateful to A. E. Gutman for subtle and revealing observations about the preliminary versions of this article.

[1]The Kantorovich Theorem [4, p. 44].

PROOF. *Sufficiency* is obvious.

Necessity: The case of $f = 0$ is trivial. If $f \neq 0$ then there is some $x \in X$ such that $f(x) \in \mathbb{R}$ and $f(x) > 0$. Denote the image $f(X)$ of X under f by R_0. Put $h := g \circ f^{-1}$, i.e., $h \in R_0^{\#}$ is the only solution for $h \circ f = g$. By hypothesis, h is a positive R-linear functional on R_0. By the Bigard Theorem [4, p. 108] h can be extended to a positive homomorphism $\bar{h} : \mathbb{R} \to \mathbb{R}$, since $R_0 - \mathbb{R}_+ = \mathbb{R}_+ - R_0 = \mathbb{R}$. Each positive automorphism of $\mathbb{R}$ is multiplication by a positive real. As the sought α we may take $\bar{h}(1)$.

The proof of the lemma is complete.

LEMMA 1.2. *Let X be an $\mathbb{R}$-seminormed vector space over some subfield R of $\mathbb{R}$. Assume that $f_1, \dots, f_N$ and g are bounded R-linear functionals on X; in symbols, $f_1, \dots, f_N, g \in X^* := L^{(m)}(X, \mathbb{R})$.*

For the inclusion

$$\{g \le 0\} \supset \bigcap_{k=1}^{N} \{f_k \le 0\}$$

to hold it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying

$$g = \sum_{k=1}^{N} \alpha_k f_k.$$

PROOF. Let us induct on N. Taking the quotient of X by the intersection of the kernels of $f_1, \dots, f_N$, we may and will assume that X is finite-dimensional over R. In other words, we have assumed that the claim is demonstrated for every collection of N functionals over each finite-dimensional space X.

To make the induction step, consider the pointwise suprema $q := f_1 \vee \dots \vee f_{N+1}$ and $p := q \vee (-g)$. Clearly, $p(x) \ge 0$ for all $x \in X$. Indeed, if one of the reals $f_k(x)$ is strictly greater than zero then so is $q(x)$. If all $f_1(x), \dots, f_{N+1}$ are negative then so is $g(x)$. Hence, $p(x) \ge -g(x) \ge 0$.

The field $\mathbb{R}$ over R admits convex analysis [4, p. 119]. Consequently, there are positive reals γ_1 and γ_2 such that $\gamma_1 + \gamma_2 = 1$ and $\gamma_1 f - \gamma_2 g = 0$ for some f belonging to the subdifferential $\partial(q)$ of q.

If $\gamma_2 > 0$ then we are done since $\partial(q) = \operatorname{co}\{f_1, \dots, f_{N+1}\}$.

If $\gamma_2 = 0$ then

$$\sum_{k=1}^{N+1} t_k f_k = 0$$

for some convex combination $t_1, \dots, t_{N+1}$ of reals. One of the coefficients $t_1, \dots, t_{N+1}$ is other than zero. For definiteness, we may and will assume that $t_{N+1} \neq 0$. Thus,

$$-f_{N+1} = \sum_{k=1}^{N} \bar{t}_k f_k$$

for some positive reals $\bar{t}_k$, $k := 1, \dots, N$.

Put $X_0 := \{f_{N+1} = 0\} = \ker(f_{N+1})$. If $x_0 \in X_0$ and $f_k(x_0) \le 0$ for all $k := 1, \dots, N$, then $g(x_0) \le 0$ by hypothesis. Therefore, by the induction assumption there are positive reals $\beta_1, \dots, \beta_N$ satisfying $h|_{X_0} = 0$ where

$$h := g - \sum_{k=1}^{N} \beta_k f_k.$$

The functionals h and f_{N+1} are bounded by hypothesis and so may be viewed as $\mathbb{R}$-linear over the completion of X. Therefore,

$$g - \sum_{k=1}^{N} \beta_k f_k = \gamma f_{N+1}$$

for some $\gamma \in \mathbb{R}$. If $\gamma \geq 0$ then we have the sought presentation for g. If $\gamma < 0$ then

$$g = \sum_{k=1}^{N} (\beta_k + |\gamma| \bar{t}_k) f_k.$$

The proof of the lemma is complete.

We will interpret Lemma 1.2 in a suitable Boolean valued universe after relevant preliminaries.

2. Boolean Models

Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to the Boolean valued models by Vopěnka, Scott, and Solovay. Scott had forecasted the area in 1969:

> *We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.*

Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis.

2.1. The Universe. Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha)\ x : \operatorname{dom}(x) \to \mathbb{B} \ \&\ \operatorname{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

with On the class of all ordinals. The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

Consider the equivalence $\{(x, y) \mid [\![x = y]\!] = \mathbb{1}\}$ on $\mathbb{V}^{(\mathbb{B})}$. Choosing a representative of the least rank in each class of equivalent functions, we come to the *separated universe* also denoted by $\mathbb{V}^{(\mathbb{B})}$.

2.2. The canonical embedding. Given $x \in \mathbb{V}$, denote by $x^\wedge$ the *standard name* of x in $\mathbb{V}^{(B)}$; i.e., the member of $\mathbb{V}^{(B)}$ that is defined by recursion as follows:

$$\varnothing^\wedge := \varnothing, \quad \operatorname{dom}(x^\wedge) := \{y^\wedge \mid y \in x\}, \quad \operatorname{im}(x^\wedge) := \{\mathbb{1}\}.$$

2.3. Descending. Given φ, a formula of ZFC, and y, an element of $\mathbb{V}^{\mathbb{B}}$; put $A_\varphi := A_{\varphi(\cdot,\ y)} := \{x \mid \varphi(x,\ y)\}$. The *descent* $A_\varphi{\downarrow}$ of a class A_φ is

$$A_\varphi{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}\ \&\ [\![\varphi(t,\ y)]\!] = \mathbb{1}\}.$$

If $t \in A_\varphi{\downarrow}$ then it is said that t *satisfies* $\varphi(\cdot,\ y)$ *inside* $\mathbb{V}^{(\mathbb{B})}$. The *descent* $x{\downarrow}$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}\ \&\ [\![t \in x]\!] = \mathbb{1}\},$$

i.e., $x{\downarrow} = A_{\cdot\in x}{\downarrow}$. The class $x{\downarrow}$ is a set. If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists t \in x{\downarrow})[\![(\exists z \in x)\ \varphi(z)]\!] = [\![\varphi(t)]\!].$$

2.4. Ascending. Assume that $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(B)}$. Put $\varnothing{\uparrow} := \varnothing$, while $\operatorname{dom}(x{\uparrow}) := x$ and $\operatorname{im}(x{\uparrow}) := \{\mathbb{1}\}$ in case $x \neq \varnothing$. The member $x{\uparrow}$ of the separated universe $\mathbb{V}^{(\mathbb{B})}$; i.e., the distinguished representative of the class $\{y \in \mathbb{V}^{(B)} \mid [\![y =\ x{\uparrow}]\!] = \mathbb{1}\}$, is the *ascent* of x.

For all $x \in \mathscr{P}(\mathbb{V}^{(B)})$ and every formula φ, the following are valid:

$$[\![(\forall z \in x{\uparrow})\,\varphi(z)]\!] = \bigwedge_{y\in x}[\![\varphi(y)]\!], \quad [\![(\exists z \in x{\uparrow})\,\varphi(z)]\!] = \bigvee_{y\in x}[\![\varphi(y)]\!].$$

Assume that $X, Y \subset \mathbb{V}^{(B)}$ and Φ is a total correspondence from X to Y, i.e., $\operatorname{dom}(\Phi) = X$. There is a unique correspondence $\Phi{\uparrow}$ from $X{\uparrow}$ to $Y{\uparrow}$ inside $\mathbb{V}^{(B)}$ such that the equality $\Phi{\uparrow}(A{\uparrow}) = \Phi(A){\uparrow}$ holds for every subset A of $\operatorname{dom}(\Phi)$ if and only if Φ is *extensional*; i.e., Φ enjoys the property

$$y_1 \in \Phi(x_1) \to [\![x_1 = x_2]\!] \leq \bigvee_{y_2\in\Phi(x_2)} [\![y_1 = y_2]\!]$$

for $x_1, x_2 \in \operatorname{dom}(\Phi)$. This $\Phi{\uparrow}$ in $\mathbb{V}^{(B)}$ is the *ascent* of Φ.

2.5. The Reals Within. There is an object $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e.,

$$[\![\mathscr{R} \text{ is the reals}]\!] = \mathbb{1}.$$

Let $\mathscr{R}{\downarrow}$ be the descent of the carrier $|\mathscr{R}|$ of the algebraic system

$$\mathscr{R} := (|\mathscr{R}|, +,\ \cdot\,, 0, 1, \leq)$$

inside $\mathbb{V}^{(\mathbb{B})}$. Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}{\downarrow}$ as follows:

$$\begin{gathered} x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1}; \\ xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1}; \\ x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1}; \\ \lambda x = y \leftrightarrow [\![\lambda^\wedge x = y]\!] = \mathbb{1} \\ (x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbb{R}). \end{gathered}$$

2.6. Gordon Theorem. *$\mathscr{R}{\downarrow}$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathscr{R}{\downarrow})$ isomorphic to $\mathbb{B}$.*

More details on Boolean valued analysis are collected in [3].

3. Simultaneous Linear Operator Inequalities

Theorem 3.1. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1,\dots,A_N$ and B belong to $L^{(m)}(X,Y)$.*

The following are equivalent:

(1) *For all $b\in\mathbb{B}$, the operator inequality $bBx\le 0$ is a consequence of the simultaneous linear operator inequalities $bA_1x\le 0,\dots,bA_Nx\le 0$, i.e.,*

$$\{bB\le 0\}\supset\{bA_1\le 0\}\cap\dots\cap\{bA_N\le 0\}.$$

(2) *There are positive orthomorphisms $\alpha_1,\dots,\alpha_N\in \operatorname{Orth}(m(Y))$ such that*

$$B=\sum_{k=1}^{N}\alpha_kA_k;$$

i.e., B lies in the operator convex conic hull of $A_1,\dots,A_N$.

Proof. (2)→ (1): If $B=\sum_{k=1}^{N}\alpha_kA_k$ for some positive $\alpha_1,\dots,\alpha_N$ in $\operatorname{Orth}(m(Y))$ while $bA_kx\le 0$ for $b\in\mathbb{B}$ and $x\in X$, then

$$bBx=b\sum_{k=1}^{N}\alpha_kA_kx=\sum_{k=1}^{N}\alpha_kbA_kx\le 0$$

since orthomorphisms commute and projections are orthomorphisms of $m(Y)$.

(1)→ (2): Consider the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over the base $\mathbb{B}$ of Y. By the Gordon Theorem the ascent $Y{\uparrow}$ of Y is $\mathscr{R}$, the reals inside $\mathbb{V}^{(\mathbb{B})}$.

Using the canonical embedding, we see that $X^{\wedge}$ is an $\mathscr{R}$-seminormed vector space over the standard name $\mathbb{R}^{\wedge}$ of the reals $\mathbb{R}$. Moreover, $\mathbb{R}^{\wedge}$ is a subfield and sublattice of $\mathscr{R}=Y{\uparrow}$ inside $\mathbb{V}^{(\mathbb{B})}$.

Put $f_k:=A_k{\uparrow}$ for all $k:=1,\dots,N$ and $g:=B{\uparrow}$. Clearly, all $f_1,\dots,f_N,g$ belong to $(X^{\wedge})^*$ inside $\mathbb{V}^{\mathbb{B}}$.

Define the finite sequence

$$f:\{1,\dots,N\}^{\wedge}\to(X^{\wedge})^*$$

as the ascent of $(f_1,\dots,f_N)$. In other words, the truth values are as follows:

$$[\![f_{k^{\wedge}}(x^{\wedge})=A_kx]\!]=\mathbb{1},\quad [\![g(x^{\wedge})=Bx]\!]=\mathbb{1}$$

for all $x\in X$ and $k:=1,\dots,N$.

Put

$$b:=[\![A_1x\le 0^{\wedge}]\!]\wedge\dots\wedge[\![A_Nx\le 0^{\wedge}]\!].$$

Then $bA_kx\le 0$ for all $k:=1,\dots,N$ and $bBx\le 0$ by (1).

Therefore,

$$[\![A_1x\le 0^{\wedge}]\!]\wedge\dots\wedge[\![A_Nx\le 0^{\wedge}]\!]\le[\![Bx\le 0^{\wedge}]\!].$$

In other words,

$$[\![(\forall k:=1^{\wedge},\dots,N^{\wedge})f_k(x^{\wedge})\le 0^{\wedge}]\!]$$
$$=\bigwedge_{k:=1,\dots,N}[\![f_{k^{\wedge}}(x^{\wedge})\le 0^{\wedge}]\!]\le[\![g(x^{\wedge})\le 0^{\wedge}]\!].$$

Using Lemma 1.2 inside $\mathbb{V}^{(\mathbb{B})}$ and appealing to the maximum principle of Boolean valued analysis, we infer that there is a finite sequence $\alpha : \{1^\wedge, \dots, N^\wedge\} \to \mathscr{R}_+$ inside $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$[\![(\forall x \in X^\wedge) g(x) = \sum_{k=1^\wedge}^{N^\wedge} \alpha(k) f_k(x)]\!] = \mathbb{1}.$$

Put $\alpha_k := \alpha(k^\wedge) \in \mathscr{R}_+{\downarrow}$ for $k := 1, \dots, N$. Multiplication by an element in $\mathscr{R}{\downarrow}$ is an orthomorphism of $m(Y)$. Moreover,

$$B = \sum_{k=1}^{N} \alpha_k A_k,$$

which completes the proof.

Lemma 1.1, describing the consequences of a single inequality, does not restrict the class of functionals under consideration. The analogous version of the Farkas Lemma simply fails for two simultaneous inequalities in general. Indeed, the inclusion $\{f = 0\} \subset \{g \leq 0\}$ equivalent to the inclusion $\{f = 0\} \subset \{g = 0\}$ does not imply that f and g are proportional in the case of an arbitrary subfield of $\mathbb{R}$. It suffices to look at $\mathbb{R}$ over the rationals $\mathbb{Q}$, take some discontinuous $\mathbb{Q}$-linear functional on $\mathbb{R}$ and the identity automorphism of $\mathbb{R}$. This gives grounds for the next result.

Theorem 3.2. *Let X be a real vector space and let Y be a Kantorovich space. Take A and B in $L(X, Y)$. The following are equivalent:*

(1) $(\exists \alpha \in m(Y))\ B = \alpha A$*;*

(2) *There is a projection $\varkappa \in \mathbb{B}$ such that*

$$\{\varkappa b B \leq 0\} \supset \{\varkappa b A \leq 0\};$$

$$\{\neg \varkappa b B \leq 0\} \supset \{\neg \varkappa b A \geq 0\}$$

for all $b \in \mathbb{B}$.

Proof. Boolean valued analysis reduces the claim to the case of the reals. Applying Lemma 1.1 twice and writing down the truth values, complete the proof.

4. Simultaneous Sublinear Inequalities

We now turn to the Farkas Lemma for sublinear operators. Denote the set of sublinear operators from X to Y by $\mathrm{Sub}(X, Y)$. An element $P \in \mathrm{Sub}(X, Y)$ is *polyhedral*, in symbols $P \in \mathrm{PSub}(X, Y)$, provided that P is the upper envelope of finitely many linear operators; i.e., there is a finite set $\Lambda \subset L(X, Y)$ such that $P(x) = P_\Lambda(x) := \sup\{Ax \mid A \in \Lambda\}$. In case X is furnished with some Y-seminorm, we consider the set of dominated sublinear operators $\mathrm{Sub}^{(m)}(X, Y)$ and the set of dominated polyhedral sublinear operators $\mathrm{PSub}^{(m)}(X, Y)$, implying the operators whose support sets lie in $L^{(m)}(X, Y)$.

We start with two lemmas in the scalar case, the second generalizing the main result of [5].

Lemma 4.1. *Let X be a real vector space. Assume that $f_1, \dots, f_N \in X^\#$ and $p \in \mathrm{Sub}(X) := \mathrm{Sub}(X, \mathbb{R})$.*

For the inclusion

$$\{p \geq 0\} \supset \bigcap_{k=1}^{N} \{f_k \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ *satisfying*

$$(\forall x \in X)\ p(x) + \sum_{k=1}^{N} \alpha_k f_k(x) \geq 0.$$

Proof. Sufficiency is obvious and we will prove necessity. To this end put

$$H := \bigcap_{k=1}^{N} \{f_k \leq 0\}.$$

Clearly, H is a (convex) cone in X. By hypothesis, $p(x) \geq 0$ for all $x \in H$. By separation (cp. [4, 3.2.16]), there is $l \in \partial(p)$ such that $l(h) \geq 0$ for all $h \in H$. By the Farkas Lemma

$$-l = \sum_{k=1}^{N} \alpha_k f_k$$

for some positive $\alpha_1, \dots, \alpha_N$, which completes the proof.

Lemma 4.2. *Let* X *be a real vector space. Assume that* $p_1, \dots, p_N \in \mathrm{PSub}(X) := \mathrm{PSub}(X, \mathbb{R})$ *and* $p \in \mathrm{Sub}(X)$.

The following are equivalent:

(1) $\{p \geq 0\} \supset \bigcap_{k=1}^{N} \{p_k \leq 0\}$;

(2) *There are* $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ *such that*

$$(\forall x \in X)\ p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \geq 0.$$

Proof. By hypothesis there are finite subsets $\Lambda_1, \dots, \Lambda_N$ of $X^{\#}$ such that $p_k = P_{\Lambda_k}$ for $k := 1, \dots, N$. Let Λ be the disjoint union of all Λ_k for $k := 1, \dots, N$. Clearly,

$$\bigcap_{k=1}^{N} \{p_k \leq 0\} = \bigcap_{\lambda \in \Lambda} \{\lambda \leq 0\}.$$

By Lemma 4.1 there are some $(\beta_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}_+$ such that for all $x \in X$ we have

$$\begin{aligned} 0 &\leq p(x) + \sum_{\lambda \in \Lambda} \beta_\lambda \lambda(x) \\ &= p(x) + \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k} \beta_\lambda \lambda(x) \\ &\leq p(x) + \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k} \beta_\lambda p_k(x) \\ &= p(x) + \sum_{k=1}^{N} \Bigg(\sum_{\lambda \in \Lambda_k} \beta_\lambda \Bigg) p_k(x). \end{aligned}$$

Putting

$$\alpha_k := \sum_{\lambda \in \Lambda_k} \beta_\lambda$$

for $k := 1, \dots, N$, we complete the proof.

We proceed now to the operator case.

Lemma 4.3. *Let X be a vector space over some subfield R of the reals $\mathbb{R}$. Assume that $f \in X^{\#}$ and $p \in \operatorname{Sub}(X)$.*

For the inclusion

$$\{p \geq 0\} \supset \{f \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha \in \mathbb{R}_+$ satisfying $(\forall x \in X)\ p(x) + \alpha f(x) \geq 0$.

Proof. We argue as in Lemma 4.1 appealing to Lemma 1.1 instead of the Farkas Lemma.

Theorem 4.1. *Let X be a real vector space, and let Y be a Kantorovich space. Assume that $A \in L(X,Y)$ and $P \in \operatorname{Sub}(X,Y)$.*

For the inclusion

$$\{bP \geq 0\} \supset \{bA \leq 0\}$$

to hold for all $b \in \mathbb{B}$ it is necessary and sufficient that there be $\alpha \in \operatorname{Orth}(m(Y))_+$ satisfying

$$(\forall x \in X)\ P(x) + \alpha Ax \geq 0.$$

Proof. The claim follows from Lemma 4.3 by Boolean valued interpretation.

Theorem 4.2. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $P_1, \dots, P_N \in \operatorname{PSub}^{(m)}(X,Y)$ and $P \in \operatorname{Sub}^{(m)}(X,Y)$.*

The following are equivalent:

(1) *For all $b \in \mathbb{B}$, the sublinear operator inequality $bP(x) \geq 0$ is a consequence of the simultaneous polyhedral sublinear operator inequalities $bP_1(x) \leq 0, \dots, bP_N(x) \leq 0$, i.e.,*

$$\{bP \geq 0\} \supset \{bP_1 \leq 0\} \cap \dots \cap \{bP_N \leq 0\}.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \operatorname{Orth}(m(Y))$ such that*

$$(\forall x \in X)\ P(x) + \sum_{k=1}^{N} \alpha_k P_k(x) \geq 0.$$

Proof. The demonstration of the claim proceeds along the lines of Lemma 4.2 on appealing to Theorem 3.1 in place of the Farkas Lemma.

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