Contents

Preface
to the English Translation

This is a concise guide to basic sections of modern functional analysis. Included are such topics as the principles of Banach and Hilbert spaces, the theory of multinormed and uniform spaces, the Riesz–Dunford holomorphic functional calculus, the Fredholm index theory, convex analysis and duality theory for locally convex spaces.

With standard provisos the presentation is self-sufficient, whereas exposing about a hundred of celebrated "named" theorems furnished with complete proofs and culminating in the Gelfand–Naĭmark–Segal construction for C^*-algebras.

The first Russian edition was printed by the Siberian Division of the "Nauka" Publishers in 1983. Since then the monograph has served as the standard textbook on functional analysis at the University of Novosibirsk.

This volume is translated from the second Russian edition printed by the Sobolev Institute of Mathematics of the Siberian Division of the Russian Academy of Sciences in 1995. It incorporates new sections on Radon measures, the Schwartz spaces of distributions, and a supplementary list of theoretical exercises and problems for drill.

This edition was typeset using $\mathcal{AMS}$-TEX, the American Mathematical Society's TEX system.

To clear my conscience completely, I also confess that $:=$ stands for the *definor*, the *assignment operator*, $\triangleleft$ marks the beginning of a (possibly empty) proof, and $\triangleright$ signifies the end of the proof.

S. Kutateladze

Preface
to the First Russian Edition

As the title implies, this book treats of functional analysis. At the turn of the century the term "functional analysis" was coined by J. Hadamard who is famous among mathematicians for the formula of the radius of convergence of a power series. The term "functional analysis" was universally accepted then as related to the calculus of variations, standing for a new direction of analysis which was intensively developed by V. Volterra, C. Arzelà, S. Pincherle, P. Levy, and other representatives of the French and Italian mathematical schools. J. Hadamard's contribution to the recent discipline should not be reduced to the invention of the word "functional" (or more precisely to the transformation of the adjective into a proper noun). J. Hadamard was fully aware of the relevance of the rising subject. Working hard, he constantly advertised problems, ideas, and methods just evolved. In particular, to one of his students, M. Fréchet, he suggested the problem of inventing something that is now generally acclaimed as the theory of metric spaces. In this connection it is worthy to indicate that neighborhoods pertinent to functional analysis in the sense of Hadamard and Volterra served as precursors to Hausdorff's famous research heralded the birth of general topology.

For the sequel it is essential to emphasize that one of the most attractive, difficult, and important sections of classical analysis, the calculus of variations, became the first source of functional analysis.

The second source of functional analysis was provided by the study directed to creating some algebraic theory for functional equations or, stated strictly, to simplifying and formalizing the manipulations of "equations in functions" and, in particular, linear integral equations. Ascending to H. Abel and J. Liouville, the theory of the latter was considerably expanded by works of I. Fredholm, K. Neumann, F. Noether, A. Poincaré, et al. The efforts of these mathematicians fertilized soil for D. Hilbert's celebrated research into quadratic forms in infinitely many variables. His ideas, developed further by F. Riesz, E. Schmidt, et al., were the immediate predecessors of the axiomatic presentation of Hilbert space theory which was undertaken and implemented by J. von Neumann and M. Stone. The resulting section of mathematics has vigorously influenced theoretical physics, first of all, quantum mechanics. In this regard it is instructive as well as entertaining to mention that both terms, "quantum" and "functional," originated in the same year, 1900.

The third major source of functional analysis encompassed Minkowski's geometric ideas. His invention, the apparatus for the finite-dimensional geometry of convex bodies, prepared the bulk of spatial notions ensuring the modern development of analysis. Elaborated by E. Helly, H. Hahn, C. Carathéodory, I. Radon, et al., the idea of convexity has eventually shaped the fundamentals of the theory of locally convex spaces. In turn, the latter has facilitated the spread of distributions and weak derivatives which were recognized by S. L. Sobolev as drastically changing all tools of mathematical physics. In the postwar years the geometric notion of convexity has conquered a new sphere of application for mathematics, viz., social sciences and especially economics. An exceptional role in this process was performed by linear programming discovered by L. V. Kantorovich.

The above synopsis of the strands of functional analysis is schematic, incomplete, and approximate (for instance, it casts aside the line of D. Bernoulli's superposition principle, the line of set functions and integration theory, the line of operational calculus, the line of finite differences and fractional derivation, the line of general analysis, and many others). These demerits notwithstanding, the three sources listed above reflect the main, and most principal, regularity: functional analysis has synthesized and promoted ideas, concepts, and methods from the classical sections of mathematics: algebra, geometry, and analysis. Therefore, although functional analysis verbatim means analysis of functions and functionals, even a superficial glance at its history gives grounds to claim that functional analysis is algebra, geometry, and analysis of functions and functionals.

A more viable and penetrating explanation for the notion of functional analysis is given by the Soviet Encyclopedic Dictionary: "Functional analysis is one of the principal branches of modern mathematics. It resulted from mutual interaction, unification, and generalization of the ideas and methods stemming from all parts of classical mathematical analysis. It is characterized by the use of concepts pertaining to various abstract spaces such as vector spaces, Hilbert spaces, etc. It finds diverse applications in modern physics, especially in quantum mechanics."

The S. Banach treatise *Theórie des Operationes Linéares*, printed half a century ago, inaugurated functional analysis as an essential activity in mathematics. Its influence on the development of mathematics is seminal: Ubiquitous and omnipresent, Banach's ideas, propounded in the book, captivate the realm of modern mathematics.

Outstanding contribution toward progress in functional analysis belongs to the renowned Soviet scientists: I. M. Gelfand, L. V. Kantorovich, M. V. Keldysh,

A. N. Kolmogorov, M. G. Kreĭn, L. A. Lyusternik, and S. L. Sobolev. The characteristic feature of the Soviet school is that its research on functional analysis is always conducted in connection with profound applied problems. The research has expanded the scope of functional analysis which becomes the prevailing language of the applications of mathematics.

The next fact is demonstrative: In 1948 considered as paradoxical was even the title of Kantorovich's insightful article *Functional Analysis and Applied Mathematics* which provided a basis for numerical mathematics of today. Whereas in 1974 S. L. Sobolev stated that "to conceive the theory of calculations without Banach spaces is as impossible as without electronic computers."

The exponential accumulation of knowledge within functional analysis is now observed alongside a sharp rise in demand for the tools and concepts of the discipline. A thus resulting conspicuous gap widens permanently between the state-of the art of analysis as such and as reflected in the literature accessible to the reading community. To alter this ominous trend is the purpose of the present book.

Preface to the Second Russian Edition

More than a decade the monograph serves as a reference book for the compulsory and optional courses in functional analysis which are delivered at Novosibirsk State University. The span of time proves that the principles of compiling the book are legitimate. The present edition is enlarged with sections addressing fundamentals of distribution theory. Theoretical exercises are supplemented and the list of references is updated. Also, the inaccuracies are improved that were mostly indicated by my colleagues.

I seize the opportunity to express my gratitude to all those who helped me in preparation of the book. My pleasant debt is to acknowledge the financial support of the Sobolev Institute of Mathematics of the Siberian Division of the Russian Academy of Sciences, the Russian Foundation for Basic Research, the International Science Foundation and the American Mathematical Society during the compilation of the second edition.

S. Kutateladze

March, 1995