ONE FUNCTIONAL-ANALYTICAL IDEA BY ALEXANDROV IN CONVEX GEOMETRY

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In memory of A. D. Alexandrov (1912-1999).

Abstract. The functional-analytical approach by A. D. Alexandrov is discussed to the Minkowski and Blaschke structures making the set of convex compact figures into a vector space. The resulting analytical possibilities are illustrated by the isometric type problems of finding convex figures separated by current hyperplanes similar to the Urysohn and double bubble problems.

A. D. Alexandrov enriched convex geometry with the technique of functional analysis and measure theory (cf. [1]). The aim of this talk is to draw attention to some extra analytical possibilities in studying by his methods the isoperimetric type problems of the theory of convex surfaces of optimal location of several figures in the cells generated by a family of hyperplanes with prescribed normals. These are called *problems with current polyhedra*. They belong to the class of extremal *problems with free boundary* and are of profound interest due to various applied problems involving optimal location of figures. As examples we can list the "convex" versions and analogs of the "double bubble" problem and similar "soap" problems (cf. [2–5] and the references therein). In this talk we discuss some model examples, one of them illustrating the possibility of introducing extra inclusion constraints in the isometric problems of the type of the internal double bubble problem with current polyhedra. This possibility was announced in [6,7].

It is well known that the classical *Minkowski duality* which identifies a convex figure $\mathfrak{x}$ in $\mathbb{R}^N$ with its *support function* $\mathfrak{x}(z) := \sup\{(x,z) \mid x \in \mathfrak{x}\}$ for $z \in \mathbb{R}^N$. Considering the members of $\mathbb{R}^N$ as singletons, we assume that $\mathbb{R}^N$ lies in the set of all compact convex subsets $\mathcal{V}_N$ of $\mathbb{R}^N$. The Minkowski duality induces in $\mathcal{V}_N$ the structure of a cone in the space $C(S_{N-1})$ of continuous functions on the Euclidean unit sphere S_{N-1}, the boundary of the unit ball $\mathfrak{z}_N$. This parametrization is the *Minkowski structure*. Addition of the support functions of convex figures amounts to passing to the algebraic sum of the latter, also called the *Minkowski addition*. It is worth observing that the *linear span* $[\mathcal{V}_N]$ of the cone $\mathcal{V}_N$ is dense in $C(S_{N-1})$.

The second parametrization, *Blaschke structure*, results from identifying the coset of translates $\{z + \mathfrak{x} \mid z \in \mathbb{R}^N\}$ of a *convex body* $\mathfrak{x}$, which is by definition

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a convex figure with nonempty interior, and the corresponding measure on the unit sphere which we call the *surface area function* of the coset of $\mathfrak{x}$ and denote by $\mu(\mathfrak{x})$. The soundness of this parametrization rests on the celebrated Alexandrov Theorem of recovering a convex surface from its surface area function. Each surface area function is an *Alexandrov measure.* So we call a positive measure on the unit sphere which is supported by no great hypersphere and which annihilates singletons. The last property of a measure is referred to as translation invariance in the theory of convex surfaces. Thus, each Alexandrov measure is a translation-invariant additive functional over the cone $\mathcal{V}_N$. The cone of positive translation-invariant measures in the dual $C'(S_{N-1})$ of $C(S_{N-1})$ is denoted by $\mathcal{A}_N$. We now agree on some preliminaries.

Given $\mathfrak{x}, \mathfrak{y} \in \mathcal{V}_N$, we let the record $\mathfrak{x} =_{\mathbb{R}^N} \mathfrak{y}$ mean that $\mathfrak{x}$ and $\mathfrak{y}$ are equal up to translation or, in other words, are translates of one another. We may say that $=_{\mathbb{R}^N}$ is the equivalence associated with the preorder $\geq_{\mathbb{R}^N}$ on $\mathcal{V}_N$ symbolizing the possibility of inserting one figure into the other by translation. Arrange the factor set $\mathcal{V}_N/\mathbb{R}^N$ which consists of the cosets of translates of the members of $\mathcal{V}_N$. Clearly, $\mathcal{V}_N/\mathbb{R}^N$ is a cone in the factor space $[\mathcal{V}_N]/\mathbb{R}^N$ of the vector space $[\mathcal{V}_N]$ by the subspace $\mathbb{R}^N$.

There is a natural bijection between $\mathcal{V}_N/\mathbb{R}^N$ and $\mathcal{A}_N$. Namely, we identify the coset of singletons with the zero measure. To the straight line segment with endpoints x and y, we assign the measure

$$|x-y|(\varepsilon_{(x-y)/|x-y|} + \varepsilon_{(y-x)/|x-y|}),$$

where $|\cdot|$ stands for the Euclidean norm and the symbol ε_z for $z \in S_{N-1}$ stands for the *Dirac measure* supported at z. If the dimension of the affine span $\mathrm{Aff}(\mathfrak{x})$ of a representative $\mathfrak{x}$ of a coset in $\mathcal{V}_N/\mathbb{R}^N$ is greater than unity, then we assume that $\mathrm{Aff}(\mathfrak{x})$ is a subspace of $\mathbb{R}^N$ and identify this class with the surface area function of $\mathfrak{x}$ in $\mathrm{Aff}(\mathfrak{x})$ which is some measure on $S_{N-1} \cap \mathrm{Aff}(\mathfrak{x})$ in this event. Extending the measure by zero to a measure on S_{N-1}, we obtain the member of $\mathcal{A}_N$ that we assign to the coset of all translates of $\mathfrak{x}$. The fact that this correspondence is one-to-one follows easily from the Alexandrov Theorem.

The vector space structure on the set of regular Borel measures induces in $\mathcal{A}_N$ and, hence, in $\mathcal{V}_N/\mathbb{R}^N$ the structure of a cone or, strictly speaking, the structure of a commutative $\mathbb{R}_+$-operator semigroup with cancellation. This structure on $\mathcal{V}_N/\mathbb{R}^N$ is called the *Blaschke structure.* Note that the sum of the surface area functions of $\mathfrak{x}$ and $\mathfrak{y}$ generates a unique class $\mathfrak{x}\#\mathfrak{y}$ which is referred to as the *Blaschke sum* of $\mathfrak{x}$ and $\mathfrak{y}$.

Let $C(S_{N-1})/\mathbb{R}^N$ stand for the factor space of $C(S_{N-1})$ by the subspace of all restrictions of linear functionals on $\mathbb{R}^N$ to S_{N-1}. Denote by $[\mathcal{A}_N]$ the space $\mathcal{A}_N - \mathcal{A}_N$ of translation-invariant measures. It is easy to see that $[\mathcal{A}_N]$ is also the linear span of the set of Alexandrov measures. The spaces $C(S_{N-1})/\mathbb{R}^N$ and $[\mathcal{A}_N]$ are set in duality by the canonical bilinear form

$$\langle f, \mu \rangle = \frac{1}{N} \int_{S_{N-1}} f d\mu \quad (f \in C(S_{N-1})/\mathbb{R}^N,\ \mu \in [\mathcal{A}_N]).$$

For $\mathfrak{x} \in \mathcal{V}_N/\mathbb{R}^N$ and $\mathfrak{y} \in \mathcal{A}_N$, the quantity $\langle \mathfrak{x}, \mathfrak{y} \rangle$ coincides with the *mixed volume* $V_1(\mathfrak{y}, \mathfrak{x})$. The space $[\mathcal{A}_N]$ is usually furnished with the weak topology induced by the above indicated duality with $C(S_{N-1})/\mathbb{R}^N$.

By the *dual* K^* of a given cone K in a vector space X in duality with another vector space Y, we mean the set of all positive linear functionals on K; i. e., $K^* := \{y \in Y \mid (\forall x \in K)\langle x, y\rangle \geq 0\}$. Recall also that to a convex subset U of X and a point $\bar{x}$ in U there corresponds the cone

$$U_{\bar{x}} := \mathrm{Fd}(U, \bar{x}) := \{h \in X \mid (\exists \alpha \geq 0)\ \bar{x} + \alpha h \in U\}$$

which is called the *cone of feasible directions* of U at $\bar{x}$. Fortunately, description is available for all dual cones we need.

1. *The dual* $\mathcal{A}_N^*$ *of* $\mathcal{A}_N$ *is the positive cone of* $C(S_{N-1})/\mathbb{R}^N$.

2. *Let* $\bar{\mathfrak{x}} \in \mathcal{A}_N$. *Then the dual* $\mathcal{A}_{N,\bar{\mathfrak{x}}}^*$ *of the cone of feasible directions of* $\mathcal{A}_N n$ *at* $\bar{\mathfrak{x}}$ *may be represented as follows*

$$\mathcal{A}_{N,\bar{\mathfrak{x}}}^* = \{f \in \mathcal{A}_N^* \mid \langle \bar{\mathfrak{x}}, f\rangle = 0\}.$$

Assume that μ and ν are positive measures on the sphere S_{N-1}. Say that μ is *linearly stronger than* ν and write $\mu \gg_{\mathbb{R}^N} \nu$ if to each decomposition of ν into the sum of finitely many positive terms $\nu = \nu_1 + \ldots + \nu_m$ there exists a decomposition of μ into the sum of finitely many terms $\mu = \mu_1 + \ldots + \mu_m$ such that $\mu_k - \nu_k \in (\mathbb{R}^N)^*$ for all $k = 1, \ldots, m$.

3. *Let* $\mathfrak{x}$ *and* $\mathfrak{y}$ *be convex figures. Then*
(1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \mathcal{V}_N^* \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;
(2) If $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ *then* $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;
(3) $\mathfrak{x} \geq_{\mathbb{R}^2} \mathfrak{y} \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^2} \mu(\mathfrak{y})$.

4. *Let* $\bar{\mathfrak{x}}$ *and* $\mathfrak{y}$ *be convex figures. Then*
(1) If $\mathfrak{y} - \bar{\mathfrak{x}} \in \mathcal{A}_{N,\bar{\mathfrak{x}}}^*$ *then* $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$;
(2) If $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \mathcal{V}_{N,\bar{\mathfrak{x}}}^*$ *then* $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$.

In the sequel we never distinguish between a convex figure, the respective coset of translates in $\mathcal{V}_N/\mathbb{R}^N$, and the corresponding measure in $\mathcal{A}_N$.

It is worth noting that the volume $V(\mathfrak{x}) := \langle \mathfrak{x}, \mathfrak{x}\rangle$ of a convex figure $\mathfrak{x}$ is a homogeneous polynomial of degree N with respect to the Minkowski structure. That is why to calculate the subdifferential of $V(\cdot)$ is an easy matter. The particular feature of the Minkowski structure is an intricate construction of the dual of the cone of compact convex sets whose description bases on the relation $\gg_{\mathbb{R}^N}$ in the space of measures $[\mathcal{A}_N]$.

Isoperimetric-type problems with subsidiary constraints on location of convex figures comprise in a sense a unique class of meaningful problems of mathematical programming which admits two essentially different parametrizations, the Minkowski and Blaschke structures. Their principal features are clearly seen from the table.

Object of Parametrization	Minkowski's Structure	Blaschke's Structure
cone of sets	$\mathcal{V}_N/\mathbb{R}^N$	$\mathcal{A}_N$
dual cone	$\mathcal{V}_N^*$	$\mathcal{A}_N^*$
positive cone	$\mathcal{A}_N^*$	$\mathcal{A}_N$
typical linear functional	$V_1(\mathfrak{z}_N, \cdot)$ (width)	$V_1(\cdot, \mathfrak{z}_N)$ (area)
concave functional (power of volume)	$V^{1/N}(\cdot)$	$V^{(N-1)/N}(\cdot)$
simplest convex program	isoperimetric problem	Urysohn's problem
operator-type constraint	inclusion of figures	inequalities on "curvatures"
Lagrange's multiplier	surface	function
differential of volume at a point $\bar{\mathfrak{x}}$ is proportional to	$V_1(\bar{\mathfrak{x}}, \cdot)$	$V_1(\cdot, \bar{\mathfrak{x}})$

This table shows that the classical isoperimetric problem is not a convex program in the Minkowski structure for $N \geq 3$. In this event a necessary optimality condition leads to a solution only under extra regularity conditions. Whereas in the Blaschke structure this problem is a convex program whose optimality criterion reads: "Each solution is a ball."

The task of choosing an appropriate parametrization for a wide class of problems is practically unstudied in general. In particular, those problems of geometry remain unsolved which combine constraints each of which is linear in one of the two vector structures on the set of convex figures. The simplest example of an unsolved "combined" problem is the internal isoperimetric problem in the space $\mathbb{R}^N$ for $N \geq 3$.

The above geometric facts make it reasonable to address the general problem of parametrizing the important classes of extremal problems of practical provenance.

In the sequel we use the following notations:

$$p : \mathfrak{x} \mapsto V^{1/N}(\mathfrak{x}) \quad (\mathfrak{x} \in \mathcal{V}_N/\mathbb{R}^N);$$
$$\widehat{p} : \mathfrak{x} \mapsto V^{(N-1)/N}(\mathfrak{x}) \quad (\mathfrak{x} \in \mathcal{A}_N).$$

The *Minkowski inequality* is thus paraphrased as $\langle \mathfrak{x}, \mathfrak{y} \rangle \geq p(\mathfrak{x})\widehat{p}(\mathfrak{y})$.

Let us illustrate the above by a version of the Urysohn problem with a constraint on the integral width.

5. External Urysohn Problem. *Among the convex figures, including $\mathfrak{x}_0$ and having integral width fixed, find a convex body of greatest volume.*

6. Optimality Test. *A feasible convex body $\bar{\mathfrak{x}}$ is a solution to Problem 5 if and only if there are a positive critical measure μ and a positive real $\bar{\alpha} \in \mathbb{R}_+$ satisfying*

(1) $\bar{\alpha}\mu(\mathfrak{z}_N) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu$;

(2) $V(\bar{\mathfrak{x}}) + \frac{1}{N}\int_{S_{N-1}} \bar{\mathfrak{x}} d\mu = \bar{\alpha} V_1(\mathfrak{z}_N, \bar{\mathfrak{x}})$;

(3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all z in the support of μ.*

If, in particular, $\mathfrak{x}_0 = \mathfrak{z}_{N-1}$ then the sought body is a *spherical lens*, that is, the intersection of two balls of the same radius; while the critical measure is the restriction of the surface area function of the ball of radius $\bar{\alpha}^{1/(N-1)}$ to the complement of the support of the lens to S_{N-1}. If $\mathfrak{x}_0 = \mathfrak{z}_1$ and $N = 3$ then our result implies that we should seek a solution in the class of the so-called spindle-shaped constant-width surfaces of revolution.

Note also that, combining the tricks of the current section, we may write down the Euler–Lagrange equations for a wide class of isoperimetric-type extremal problems. In particular events, these are reasonable to apply together with another technique of geometry and mathematical programming. To illustrate this, we exhibit a rather typical example:

7. Urysohn Type Problem. *Among convex figures of fixed thickness and integral width, find a convex body of greatest volume.*

Recall that the *thickness* $\Delta(\mathfrak{x})$ of a convex figure $\mathfrak{x}$ is defined as follows:

$$\Delta(\mathfrak{x}) := \inf_{z \in S_{N-1}} (\mathfrak{x}(z) + \mathfrak{x}(-z)).$$

Observe first that Problem 7 is stated as "convex on the wrong side." However, applying the Minkowski symmetrization once, we see that a solution belongs to the class of centrally symmetric convex figures for which the restricted thickness may be rewritten as inclusion-type constraint.

8. Optimality Test. *Let a positive measure μ and reals $\bar{\alpha}$, $\bar{\beta} \in \mathbb{R}_+$ satisfy the following conditions:*

(1) $\bar{\alpha}\mu(\mathfrak{z}_N) + \bar{\beta}(\varepsilon_{z_0} + \varepsilon_{-z_0}) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu$;

(2) $V(\bar{\mathfrak{x}}) + \frac{1}{N}\int_{S_{N-1}} \bar{\mathfrak{x}} d\mu = \bar{\alpha} V_1(\mathfrak{z}_N, \bar{\mathfrak{x}}) + \frac{1}{N}\bar{\beta}(\bar{\mathfrak{x}}(z_0) + \bar{\mathfrak{x}}(-z_0))$;

(3) $\bar{\mathfrak{x}}(z) = \frac{1}{2}\Delta$ *for all z in the support of μ.*

Then a feasible convex body $\bar{\mathfrak{x}}$ is a solution to Problem 7.

Therefore, a convex figure $\bar{\alpha}\mathfrak{z}_N \# \bar{\beta}\mathfrak{z}_{N-1}$ of given integral width and thickness is optimal for Problem 7. In the case $N = 3$, a solution belongs to the class of the so-called cheese-shaped constant-width surfaces of revolution.

9. Internal Urysohn Problem with a Current Hyperplane. *Find two convex figures $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ lying in a given convex body $\mathfrak{x}_0$, separated by a hyperplane with the unit outer normal z_0, and having the greatest total volume of $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ given the sum of their integral widths.*

10. Optimality Test. *A feasible pair of convex bodies $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ solves Problem 9 if and only if there are convex figures $\mathfrak{x}$ and $\mathfrak{y}$ and positive reals $\bar{\alpha}$ and $\bar{\beta}$ satisfying*

(1) $\bar{\mathfrak{x}} = \mathfrak{x} + \bar{\alpha}\mathfrak{z}_2$;

(2) $\bar{\mathfrak{y}} = \mathfrak{y} + \bar{\alpha}\mathfrak{z}_2$;

(3) $\mu(\mathfrak{x}) \geq \bar{\beta}\varepsilon_{z_0}$, $\mu(\mathfrak{y}) \geq \bar{\beta}\varepsilon_{-z_0}$;

(4) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x}) \setminus \{z_0\}$;

(5) $\bar{\mathfrak{y}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x}) \setminus \{-z_0\}$*,*
with $\operatorname{supp}(\mathfrak{x})$ *standing for the support of* $\mathfrak{x}$*, i.e. the support of the surface area measure* $\mu(\mathfrak{x})$ *of* $\mathfrak{x}$*.*

Proof. The problem under consideration may be clearly rephrased as follows

$$\begin{gathered}\mathfrak{x} \le \mathfrak{x}_0;\\ \mathfrak{x}(z_0) + \mathfrak{y}(-z_0) \le 0;\\ \langle \mathfrak{x}, \mathfrak{z}_N\rangle + \langle \mathfrak{y}, \mathfrak{z}_N\rangle = \langle \bar{\mathfrak{x}}, \mathfrak{z}_N\rangle + \langle \bar{\mathfrak{y}}, \mathfrak{z}_N\rangle;\\ p(\mathfrak{x}) + p(\mathfrak{y}) \to \max\end{gathered}$$

Indeed, it suffices to note that the separation by a hyperplane with the outer normal z_0 may be written down as

$$\mathfrak{x}(z_0) = \sup\{(x, z_0) \mid x \in \mathfrak{x}\} \le \inf\{(x, z_0) \mid x \in \mathfrak{y}\} = -\mathfrak{y}(-z_0).$$

We are left with using the subdifferential optimality test for the above convex program: For feasible convex bodies $\mathfrak{x}$ and $\mathfrak{y}$ to be an optimal solution it is necessary and sufficient that there exist positive reals $\bar\alpha$ and $\bar\beta$ and measures μ and ν satisfying

$$\begin{gathered}\mu + \nu + \bar\alpha\mu(\mathfrak{z}_N) + \bar\beta(\varepsilon_{z_0} + \varepsilon_{-z_0}) - \mu(\bar{\mathfrak{x}}) - \mu(\bar{\mathfrak{y}}) \in \mathcal{V}^*_{N,\bar{\mathfrak{x}}} \times \mathcal{V}^*_{N,\bar{\mathfrak{y}}};\\ \mu(\mathfrak{x}_0 - \bar{\mathfrak{x}}) = 0; \quad \nu(\mathfrak{x}_0 - \bar{\mathfrak{y}}) = 0.\end{gathered}$$

Obviously $\mu + \bar\beta\varepsilon_{z_0}$ and $\nu + \bar\beta\varepsilon_{-z_0}$ are the surface area measures of some figures $\mathfrak{x}$ and $\mathfrak{y}$ possessing the needed properties. We are done on recalling the structure of the duals of the cone of feasible directions.

Remark 1. If we replace the condition on the integral width by a constraint on the surface area or other mixed volumes of a more general shape then we come to possibly nonconvex programs for which a similar reasoning yields the necessary extremum conditions in general. In this event the Minkowski sum must be replaced with the Blaschke sum. Therefore, some positive Blaschke-linear combination of the ball and a tetrahedron is proportional to the soltion of the internal Urysohn problem in this tetrahedron (cf. [3]).

Remark 2. The above analysis together with the Schwarz symmetrization shows in particular that the solution of the “internal” double bubble problem within the ball in the class of the union of pairs of convex figures is given by an appropriate part of the union of some spherical caps.

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