ON AN INEQUALITY OF BIEBERBACH

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A method for analyzing certain extremal problems in the theory of convex surfaces was proposed in [1]. The set \( \mathfrak{S}_n \) of convex compact subsets of an \( n \)-dimensional space \( \mathbb{R}^n \) equipped with Minkowski operations was considered as a cone in the vector space of A. G. Pinsker and extremal problems were analyzed by using mathematical programming. In particular, for the problem of maximizing a mixed volume with linear constraints it was shown how the solution of the problem produces an inequality connecting the function being maximized and the constraints. In this note, a generalized inequality of Bieberbach is obtained by a similar method. The derivation given here is based on an inequality of A. D. Aleksandrov.

We agree to denote a convex compactum and its support function by the same symbol. Let \( z_n \) be the unit sphere with center at the origin in the space \( \mathbb{R}^n \); the sphere of directions (the surface of the unit sphere) is denoted by \( Z_n \). We use the usual notation for mixed volumes \( V_{m,k} \) and mixed surface functions \( \mu_{m,k} \), i.e.,

\[
V_{m,k}(A, x, B) = V(A_1, \ldots, A_{n-m}, x, \ldots, x, B, \ldots, B)
\]

\[
= \frac{1}{n} \int_{Z_n} x d\mu(A_1, \ldots, A_{n-m}, x, \ldots, x, B, \ldots, B) = \frac{1}{n} \int_{Z_n} x d\mu_{m,k}(A, x, B).
\]

We let \( d(x) \) represent the diameter \( x \in \mathfrak{S}_n \), i.e.,

\[
d(x) = \max_{u \in Z_n} (x(u) + x(-u)) = \max_{u \in Z_n} b(x, u).
\]

Since \( b(x, u) \) (the width of \( x \) in the direction \( u \)) is a linear functional in Pinsker space, \( d(x) \) is a sublinear functional. We define the set \( \mathfrak{U}_n \) as \( \{ u \in Z_n : b(x, u) = d(x) \} \) and the cone of feasible directions

\[
\mathfrak{D}_n = \{ g \in C(Z_n) : \exists a_0 > 0: x + ag \in \mathfrak{S}_n; \{ 0 \leq a \leq a_0 \} \};
\]

here \( C(Q) \) is the space of continuous functions on the compactum \( Q \).

We shall now consider the problem of maximizing a mixed volume on the set of convex surfaces of given diameter.

Problem 1. We seek an \( x \in \mathfrak{S}_n \) from the following conditions: 1) \( d(x) = d(\tilde{x}) \); 2) \( V_{m,k}(A, x, B) \) attains a maximum. (The convex compacta \( \tilde{x}, A_1, \ldots, A_{n-m}, B \) are assumed to be solid.)

THEOREM. The following assertions are equivalent:

1) the convex compactum \( \tilde{x} \) is a solution of Problem 1;

2) for every convex compactum \( x \in \mathfrak{S}_n \), the inequality

\[
V_{m,k}(A, x, B) d^{m-1}(\tilde{x}) - V_{m,k}(A, \tilde{x}, B) d^{m-1}(x) \leq 0
\]

is satisfied. (The equality sign holds only for solutions* of Problem 1);

3) for every function \( g \) in the cone of feasible directions \( \mathfrak{D}_{m,n} \), there is satisfied the inequality

\[
\max_{u \in \mathfrak{U}_n} b(x, u) \int_{Z_n} g d\mu_{m,k}(A, x, B) \leq \max_{u \in \mathfrak{U}_n} b(g, u) \int_{Z_n} x d\mu_{m,k}(A, x, B).
\]

*More exactly, for solids that are homothetic solutions.

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Proof. Consider Problem 2: find \( x \in \mathfrak{B} \), such that 1) \( d(x) \leq d(\bar{x}) \); 2) \( G(x) = V_{m-k}^m (A, \bar{x}, B)^{1/m-k} \cdot (A, x, B) \) attains a maximum.

Thus, every solution of Problem 2 is a solution of Problem 1, and vice versa. On the other hand, Problem 2 is a problem in concave programming, since \( \mathfrak{B} \) is a convex closed cone in the space \( C(Z_n) \), and \( d(x) \) and \( G(x) \) are, respectively, sublinear and superlinear functionals of this cone. It is clear that the condition of Slater is satisfied in this problem. According to the Kuhn–Tucker theorem, \( \bar{x} \) is a solution if and only if there exists a positive \( \alpha \) such that the function

\[
\Psi(x) = G(x) + \alpha (d(\bar{x}) - d(x))
\]

attains a maximum on \( \mathfrak{B} \) at the point \( \bar{x} \). A necessary and sufficient condition for maximality, by virtue of the concavity of \( \Psi(x) \), can be written in the form of inequalities in derivatives with respect to feasible directions:

\[
\frac{\partial \Psi}{\partial z}(g) \leq 0 \quad (g \in \mathfrak{B} \setminus \mathfrak{B}),
\]

\[
\frac{\partial \Psi}{\partial x}(\bar{x}) = 0.
\]

Since the diameter \( d(x) \) is sublinear, its derivative with respect to directions can be written (see [2]) in the form

\[
d_x'(g) = \max_{u \in U_x} b(g, u).
\]

Using the last formula and the formulas for differentiation of mixed volumes given in [1], we find that (3) is equivalent to the condition \( \bar{\alpha} = V_{m-k}^m (A, \bar{x}, B)/d(\bar{x}) \). Considering (1) and (2), we obtain the required equivalences.

COROLLARY 1. Let \( \tilde{x}, A_1, \ldots, A_{n-m}, B \) be centrally symmetric convex surfaces, with

\[
\mu_{m-k} (A, \tilde{x}, B) (Z_n \setminus U) = 0.
\]

Then \( \tilde{x} \) is a solution of Problem 1.

COROLLARY 2. Let \( A_1, \ldots, A_{n-m}, B \) be centrally symmetric convex surfaces. Then for every convex surface \( x \), we have the inequality

\[
V_{m-k}^m (A, x, B) d_{m-k}^m (z) - V_{m-k}^m (A, z_n, B) d_{m-k}^m (x) \leq 0,
\]

where equality holds only for solutions of Problem 1.

COROLLARY 3. We have Bieberbach's inequality (see [3])

\[
V(x) \leq 2^{-n} V(z_n) d^m(x),
\]

where equality holds only for a sphere.

Remark. Consider the convex \( Y_n \) symmetric with respect to the origin bounding the compactum \( Y_n \), and let \( d_{y_n} (x) = \max_{u \in y_n} b(x, u) \) be the diameter of \( x \) in the metric generated by the normalizing function \( y_n \).

Using the natural isomorphism of \( C(Y_n) \) and \( C(Z_n) \), we can easily apply the foregoing reasoning to the case of the diameter \( d_{y_n} \). In particular, for centrally symmetric convex surfaces \( A_1, \ldots, A_{n-m}, B \) and for every convex surface \( x \), we obtain the inequality

\[
V_{m-k}^m (A, x, B) d_{y_n}^m (y_n) - V_{m-k}^m (A, y_n, B) d_{y_n}^m (x) \leq 0.
\]

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